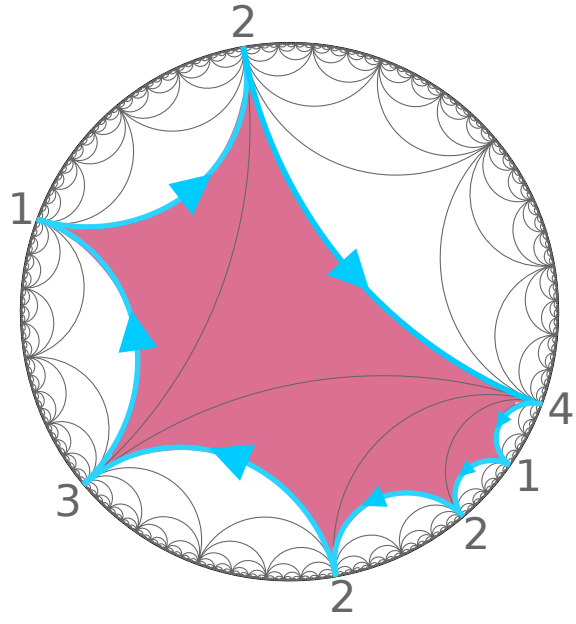
Navigating by numbers

Ian Short and Margaret Stanier



Workshop activities: https://student.desmos.com/join/8k8baa
Open the activities in another tab. Use any sensible name to join.
Make sure you have a pen and some scrap paper.

Coxeter's frieze patterns

$$
\begin{array}{ccccccccccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & & \\
 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & \\
\ldots & & & & & & & & & & & & & & & & & & & & & & & & \ldots \\
 & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & & \\
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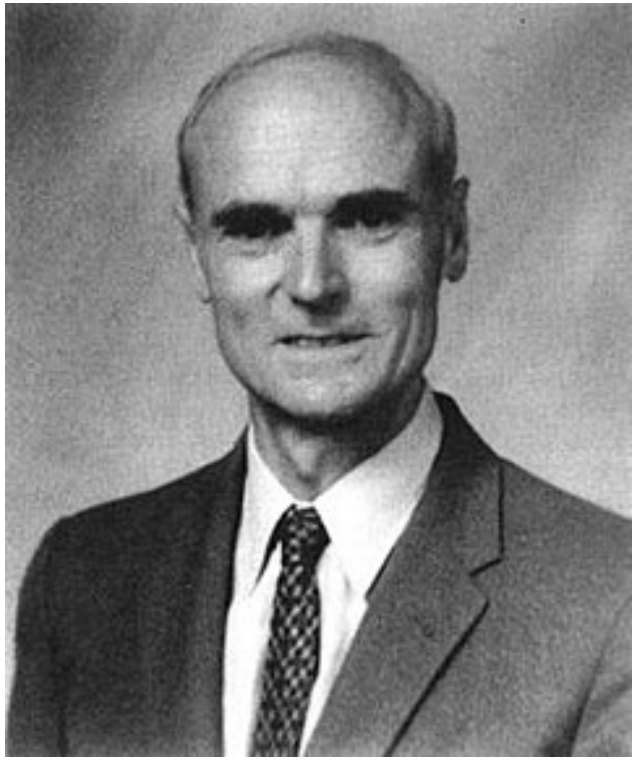
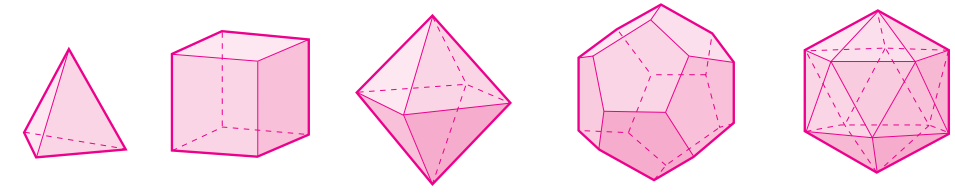
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Donald Coxeter (1907–2003)



British–Canadian mathematician whose work inspired the rise of geometry



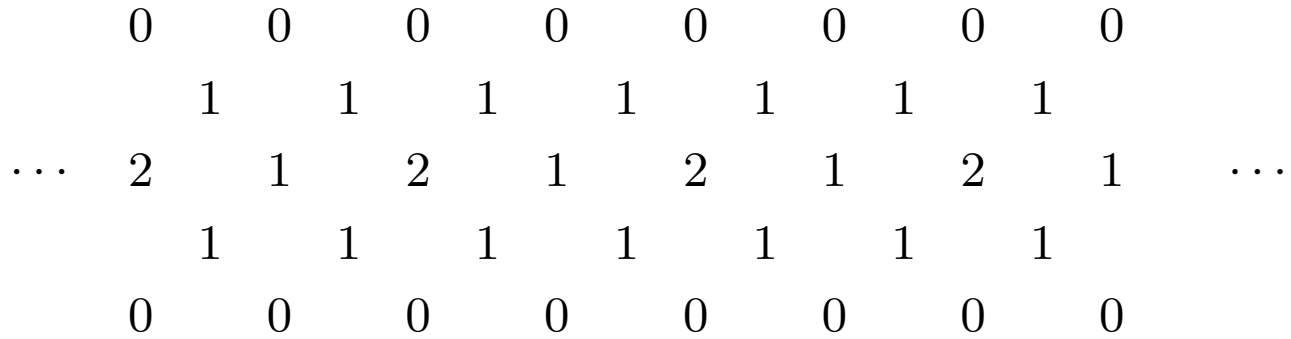
Coxeter's frieze patterns

The Open University

$$
\begin{array}{ccccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
\cdots & & 2 & & 1 & & 2 & & 1 & & 2 & & 1 & & 2 & & 1 & & \cdots \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

H.S.M. Coxeter, *Frieze patterns*, 1971

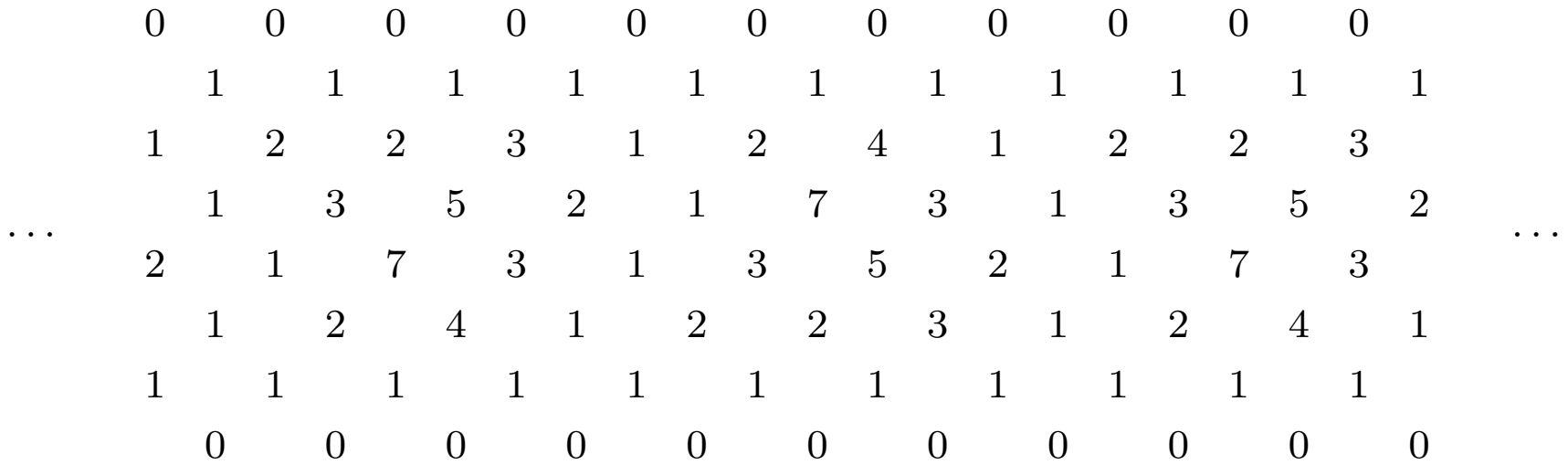
Coxeter's frieze patterns



$$\begin{array}{ccccc} & & 1 & & \\ 2 & & & & 1 \\ & & 1 & & \end{array} \qquad\qquad (2 \times 1) - (1 \times 1) = 1$$

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's frieze patterns



Coxeter's frieze patterns

The Open University

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 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 3 & \\
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\cdots & & & & & & & & & & & & & & & & & & & & \cdots \\
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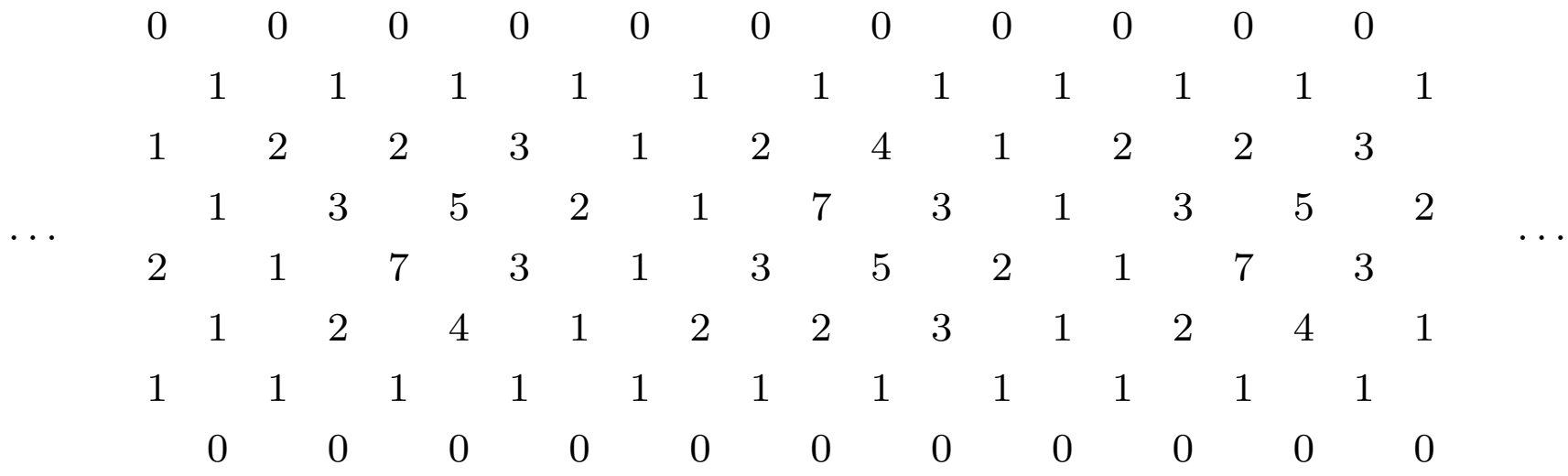
\end{array}
$$

$$
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
\qquad\qquad ad - bc = 1
$$

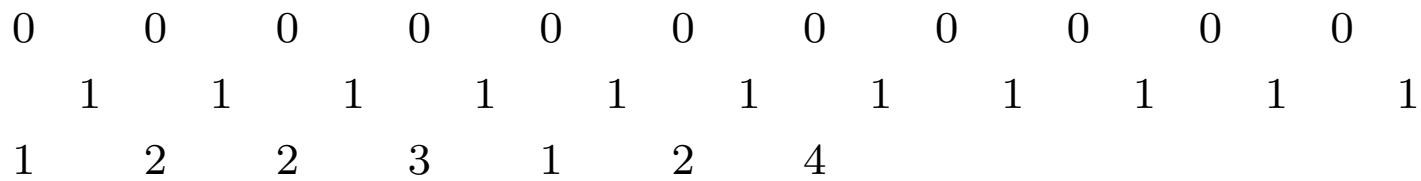
Frieze pattern codes



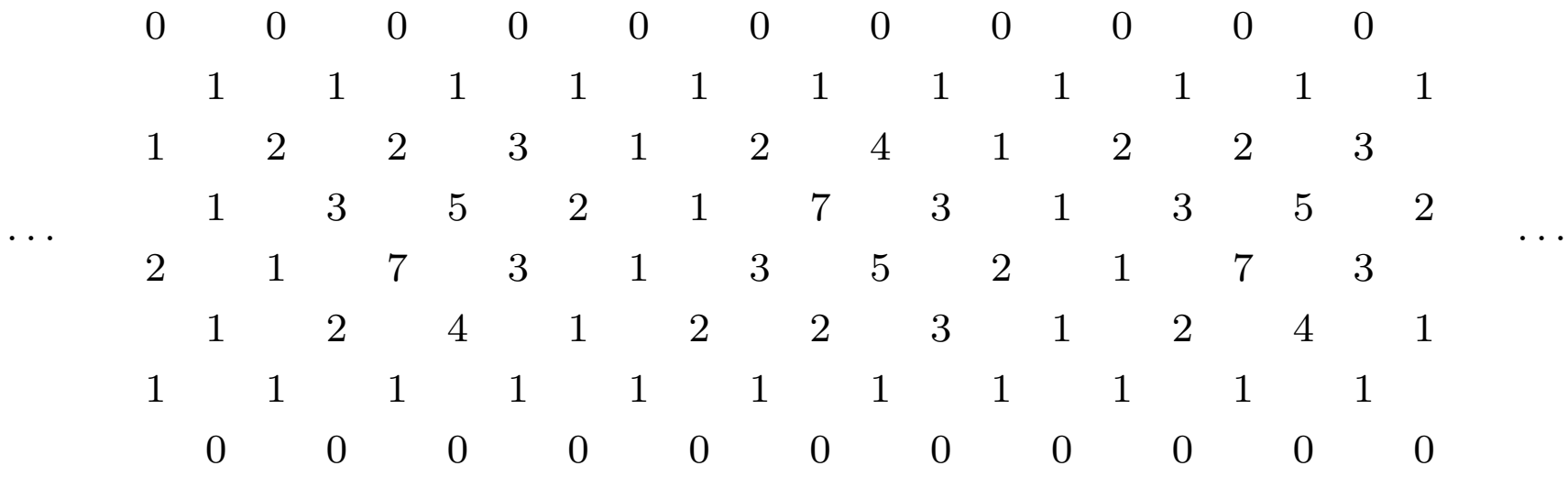
Coxeter's observation All frieze patterns are periodic.

The periodic part of the third row is called the **frieze pattern code**.

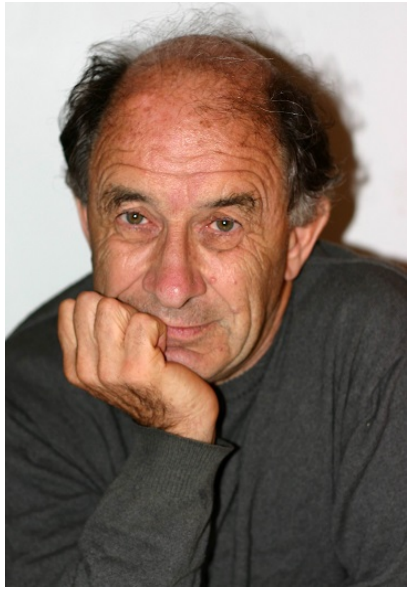
Building frieze patterns from a frieze pattern code



Building frieze patterns from a frieze pattern code



Arnold's mathematical experiments



Vladimir Arnold (1937–2010)

Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.

Activity 1

Exploring frieze patterns

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\cdots & & 2 & & 1 & & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
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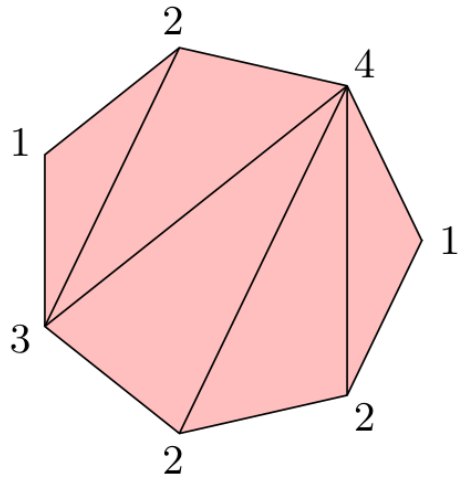
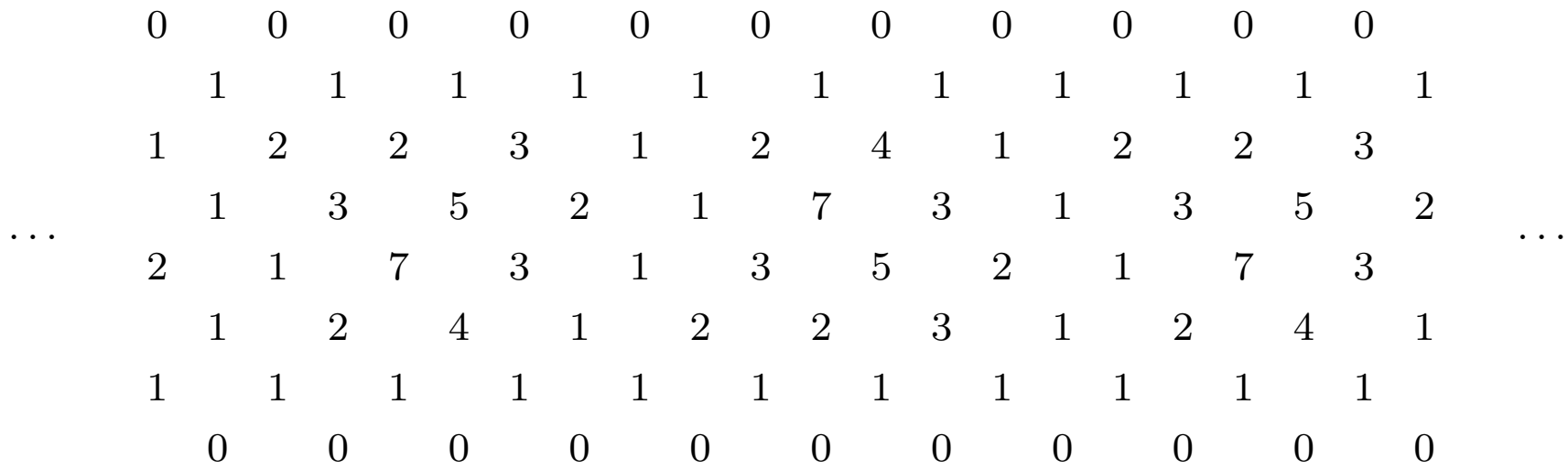
Triangulated polygons



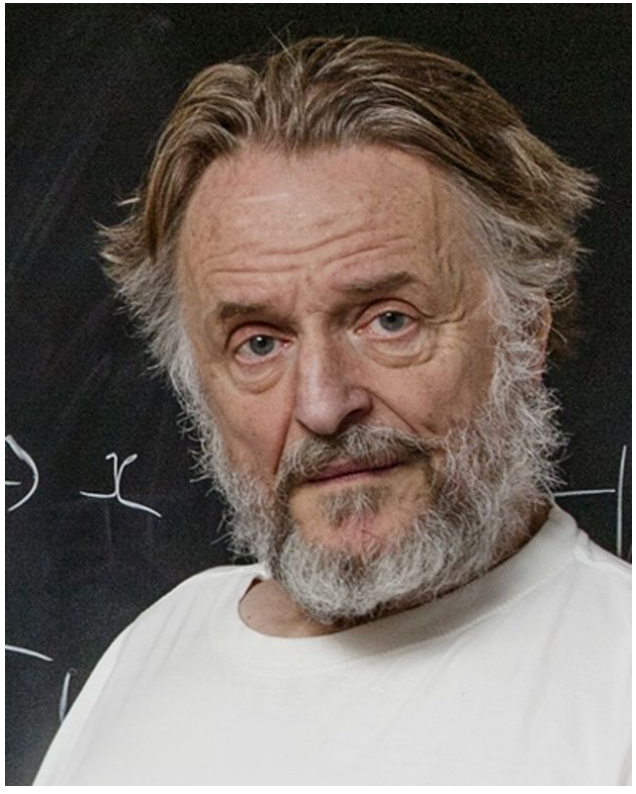
Coxeter's challenge



Coxeter's challenge Characterise frieze pattern codes.

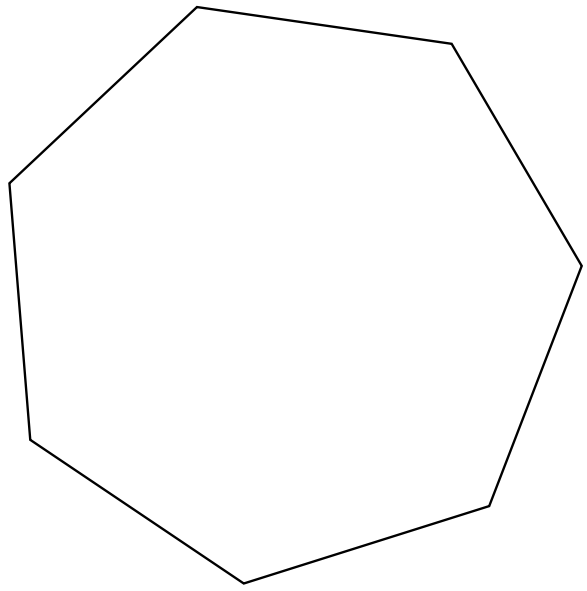
H.S.M. Coxeter, *Frieze patterns*, 1971

John Conway (1937–2020)

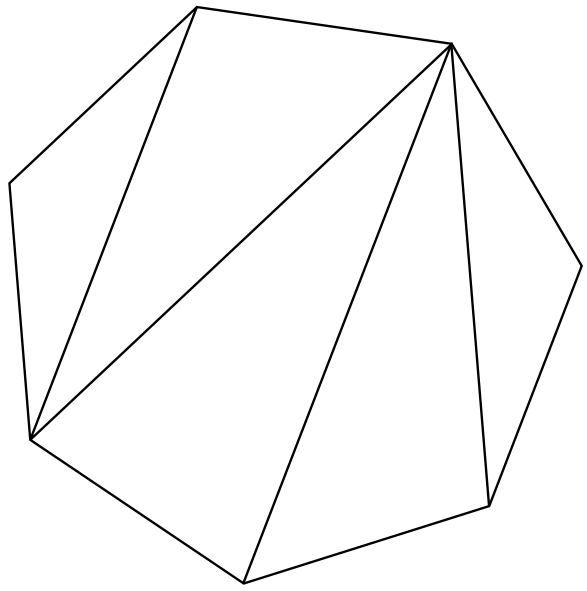


I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games IS mathematics.

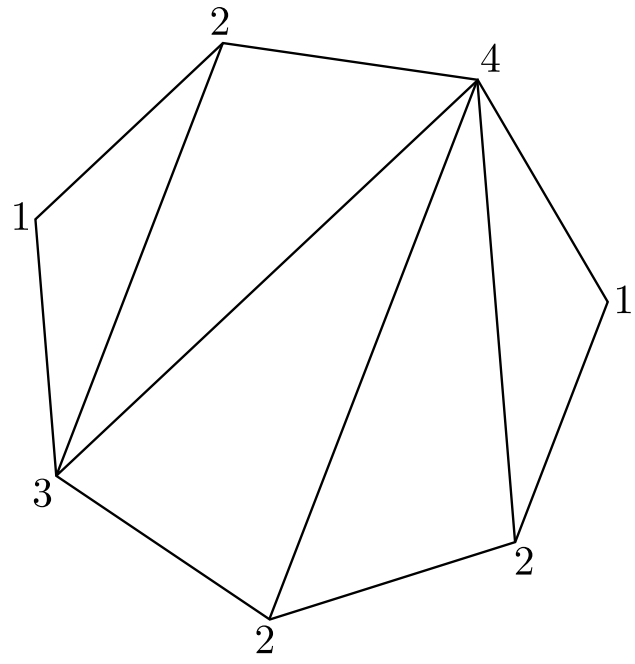
Triangulated polygons



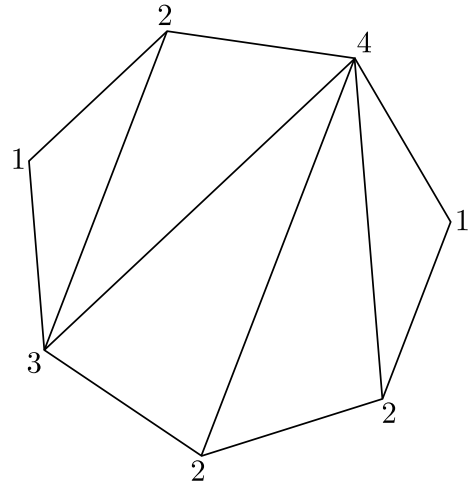
Triangulated polygons



Triangulated polygons



Conway's insight



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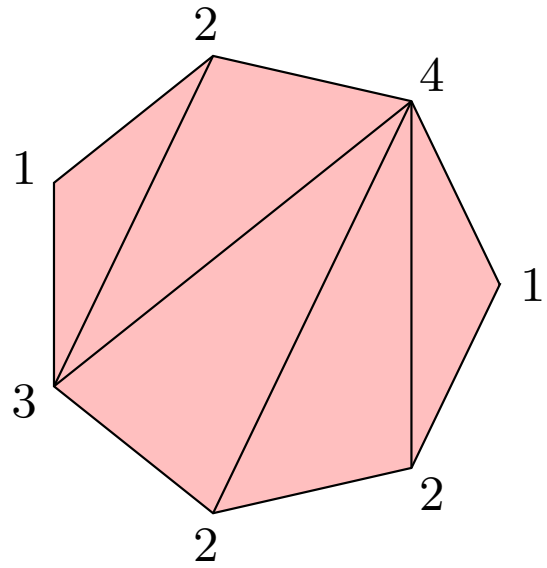
 & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 \\
\ldots & & & & & & & & & & & & & & & & & & & & \ldots \\
2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 \\
 & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
\end{array}
$$

J.H. Conway & H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, 1973

Activity 2

Constructing frieze patterns using triangulated polygons



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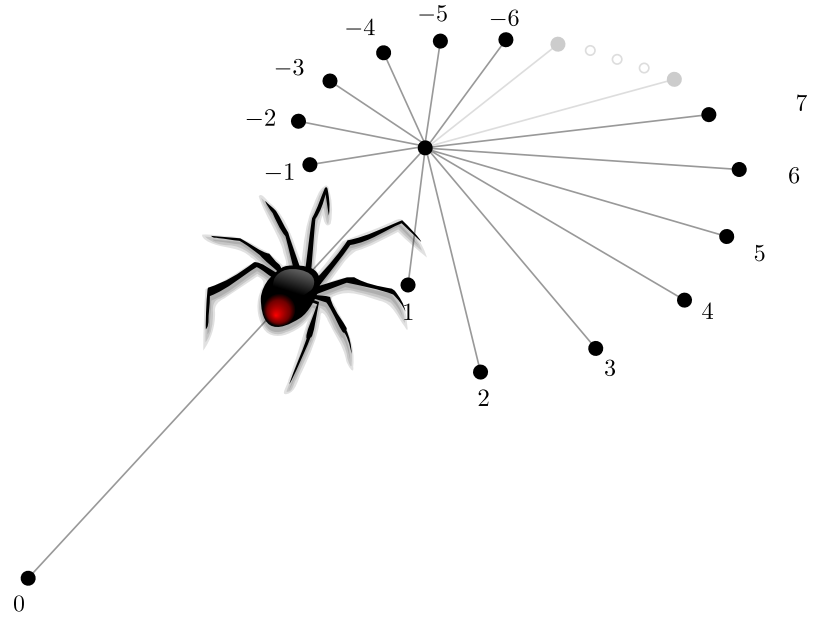
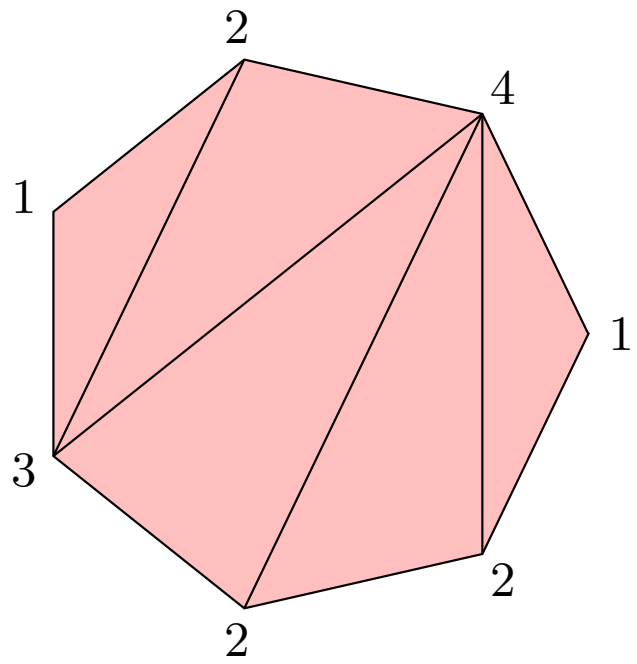
Frieze patterns with negative integers

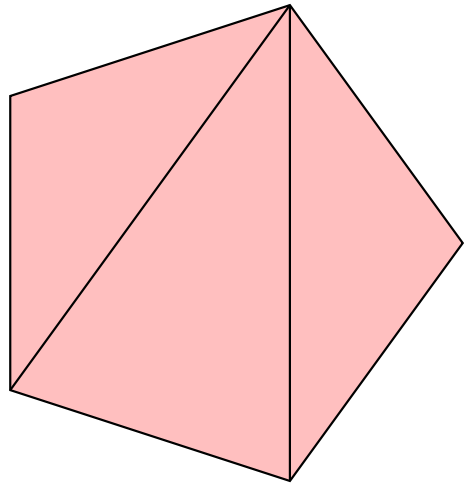


Replacing polygons by paths

The Open University



Frieze patterns with negative integers



$$
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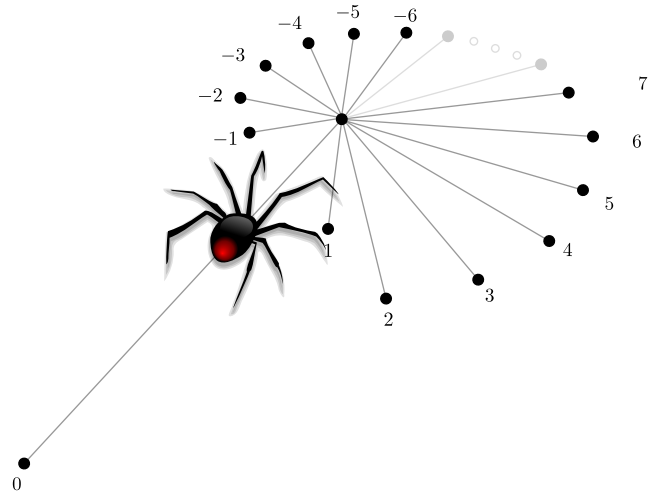
 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & 3 & & 1 & & 1 & & -1 & & 1 & & 1 & & 3 & & 1 & & 1 & & -1 & \\
\cdots\ 2 & & 2 & & 0 & & -2 & & -2 & & 0 & & 2 & & 2 & & 0 & & -2 & & -2 & \cdots \\
 & 1 & & -1 & & -1 & & -3 & & -1 & & -1 & & 1 & & -1 & & -1 & & -3 & & -1 \\
 -1 & & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Ian Short, *Classifying SL_2-tilings*, 2021

The Open University

Activity 3

Frieze patterns with negative integers



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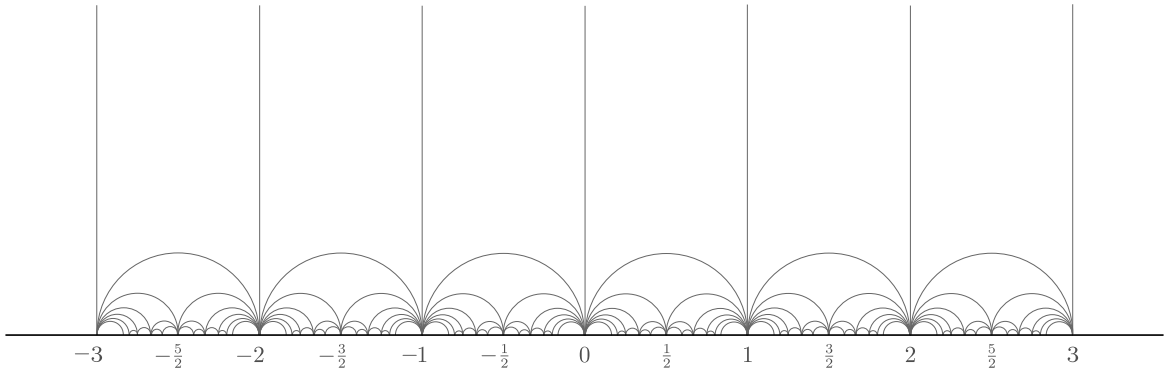
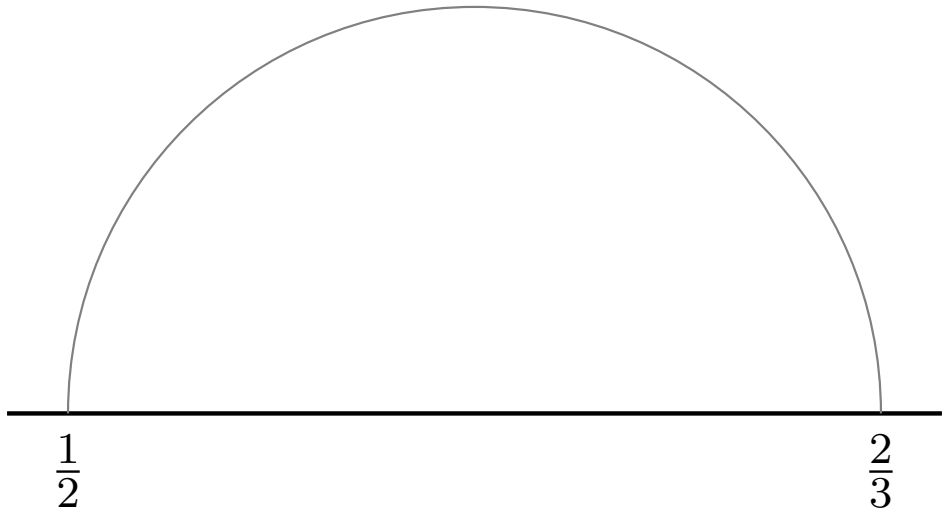
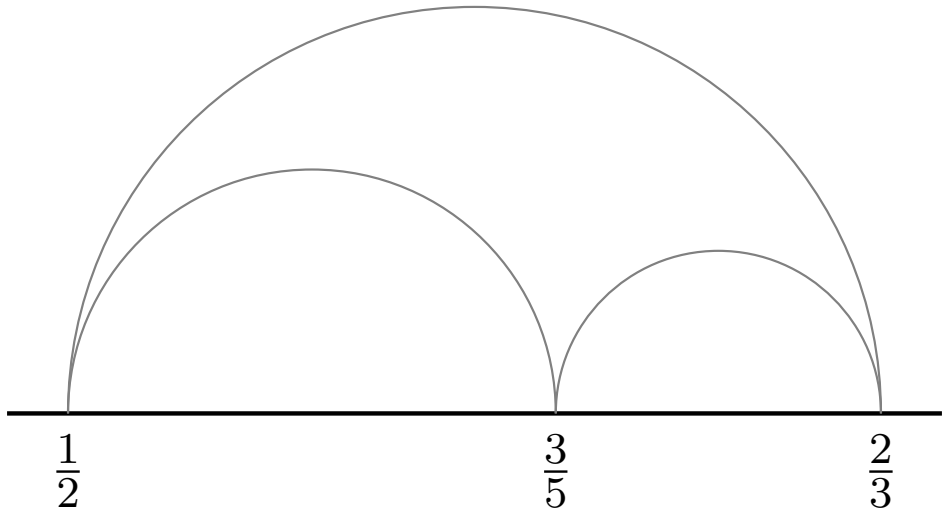
The Farey tessellation



Farey mediant



Farey mediant



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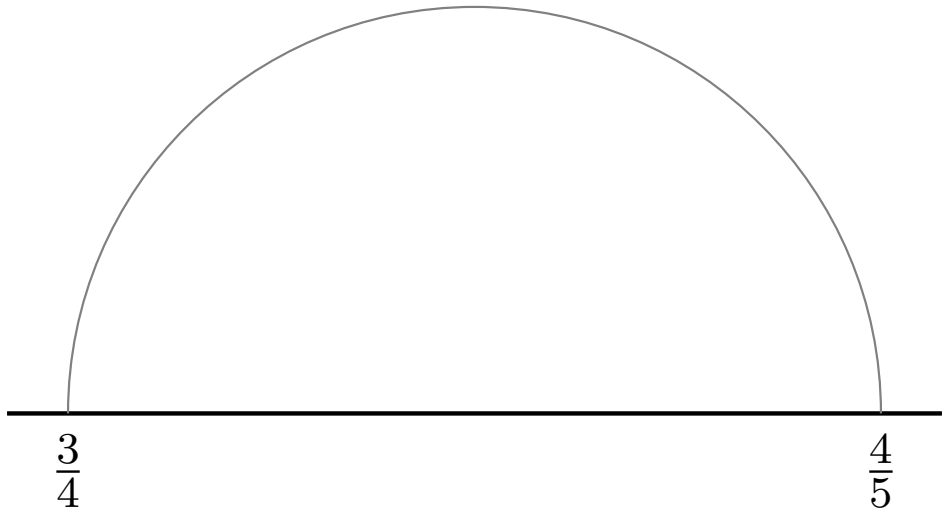

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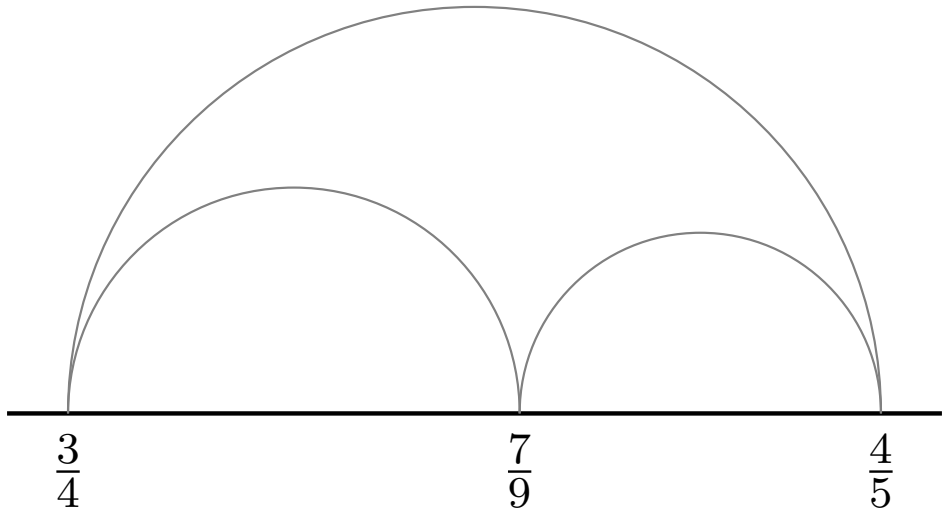

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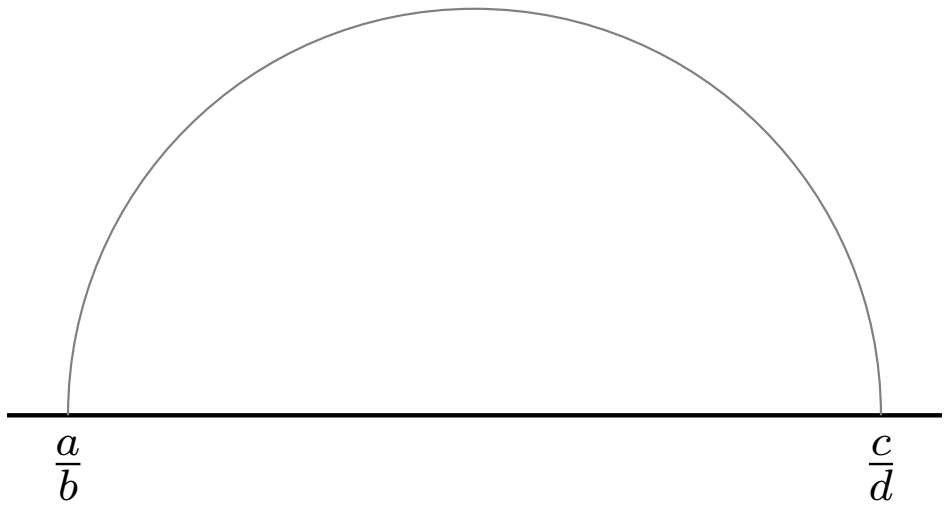

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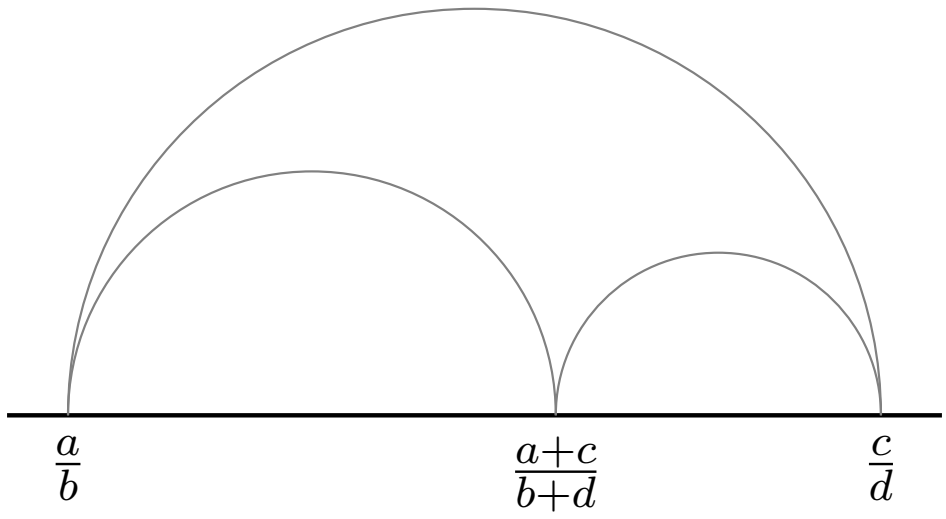
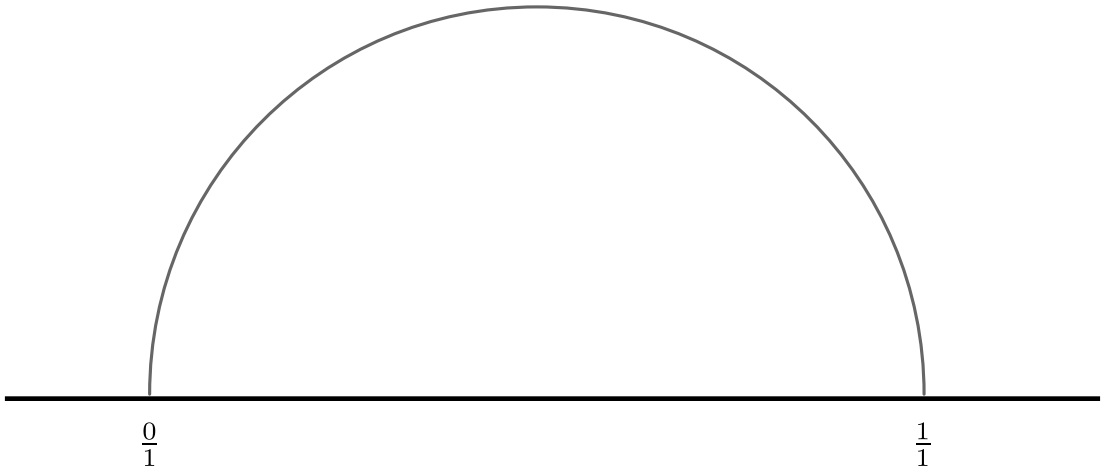


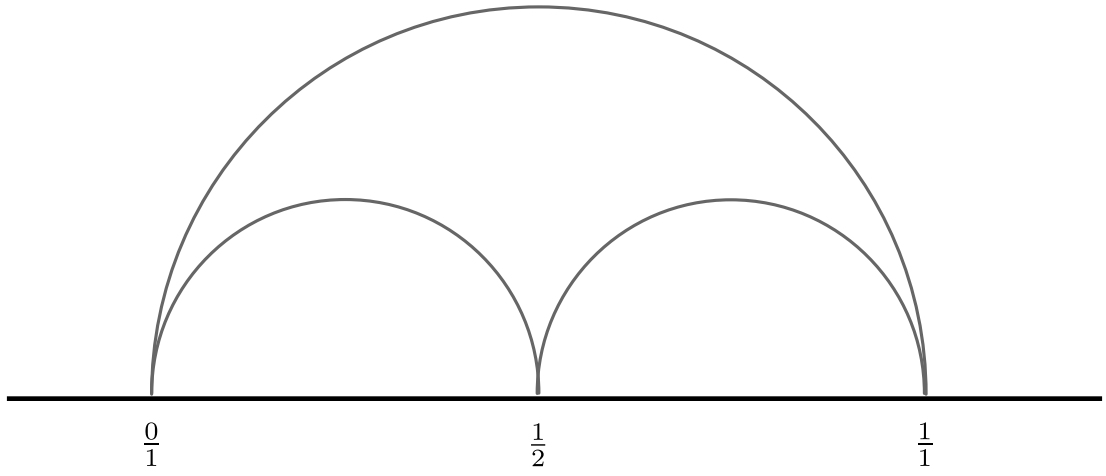
Repeated Farey mediants

The Open University

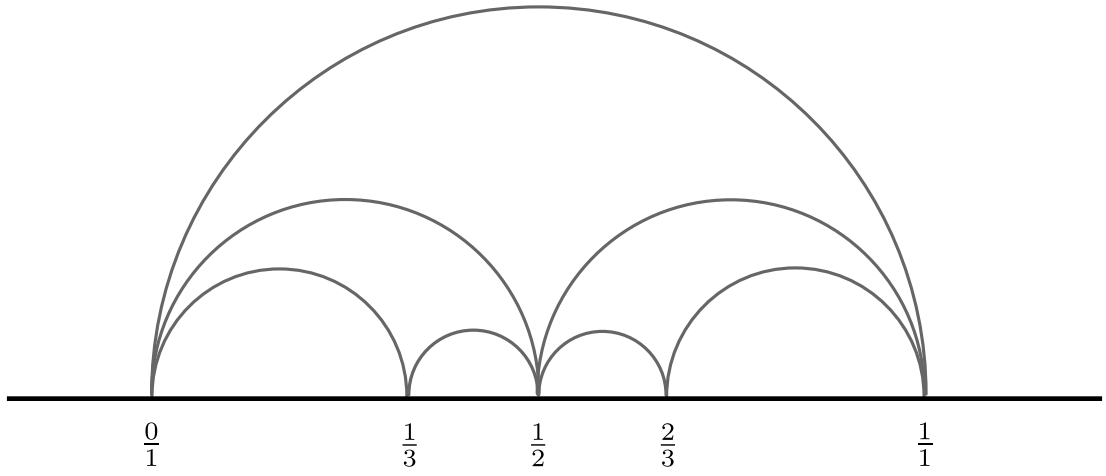


Repeated Farey mediants

The Open University



Repeated Farey mediants



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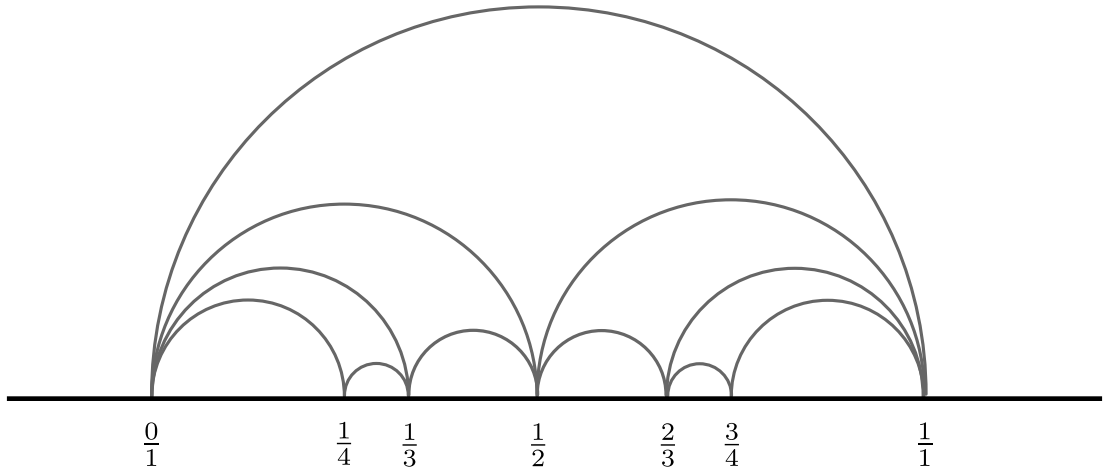
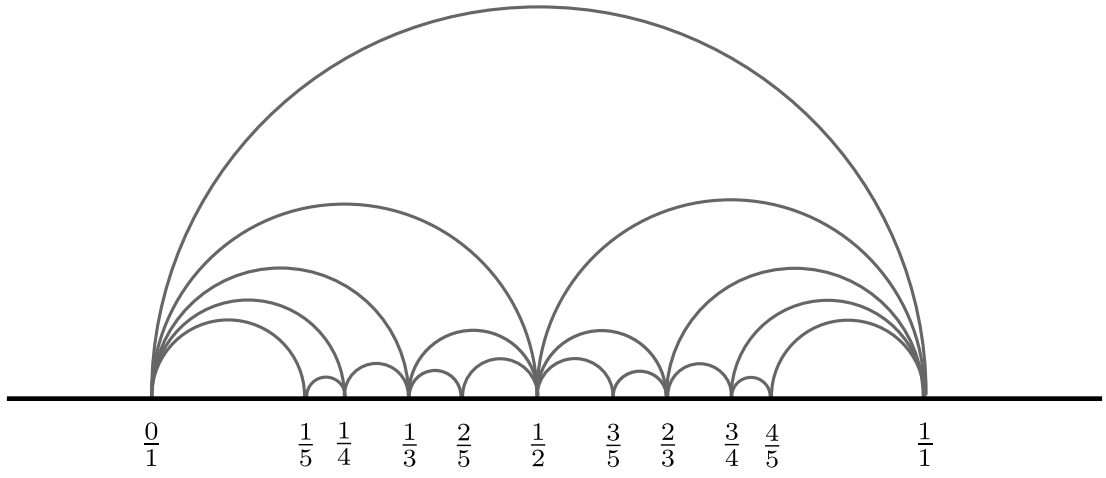


Repeated Farey mediants



Farey's fractions

the largest denominator, then, in a part of the arranged Table, we should have $\frac{15}{52}$, $\frac{28}{97}$, $\frac{13}{45}$, $\frac{24}{83}$, $\frac{11}{38}$, &c.; and if the third of these fractions be given, we have $\frac{15+13}{52+45} = \frac{28}{97}$ the second: or $\frac{13+11}{45+38} = \frac{24}{83}$ the fourth of them: and so in all the other cases.

I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

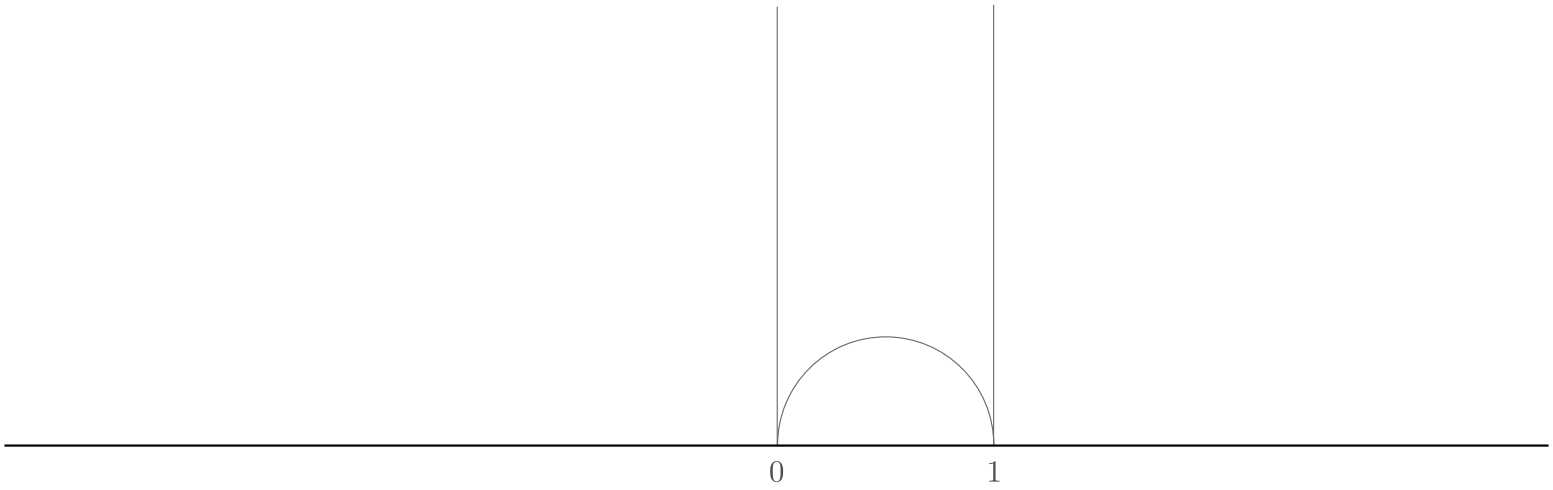
Sir,

Your obedient humble servant,

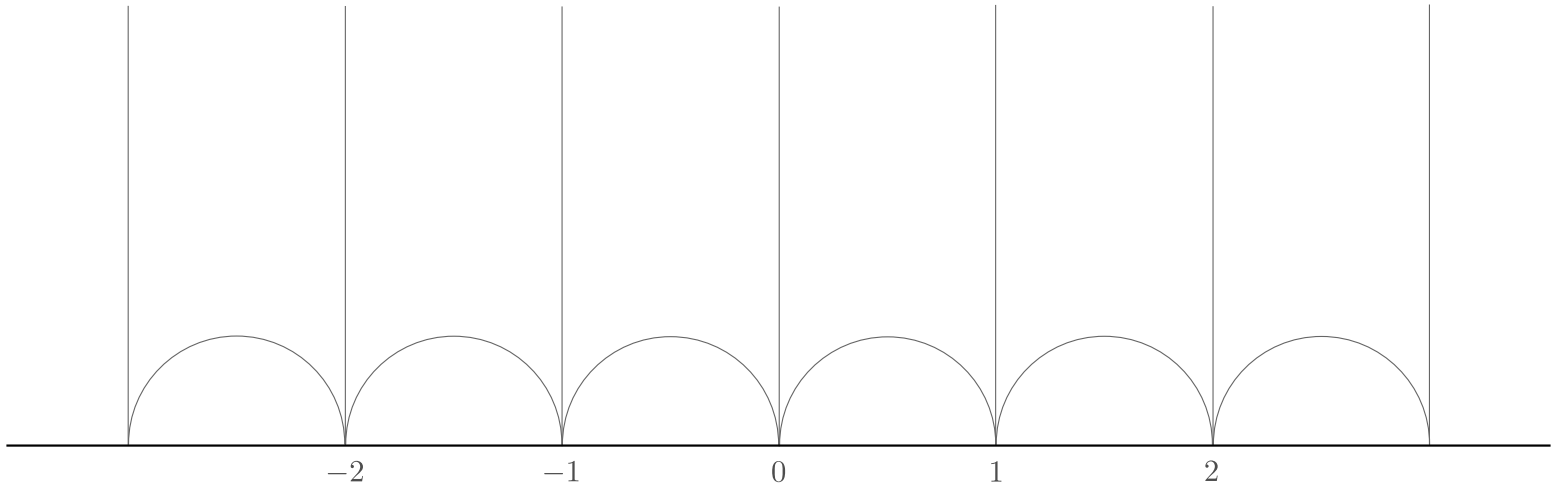
Howland-street. J. FAREY.

John Farey (1766–1826)
On a curious property of fractions
Philosphical Magazine, 1816

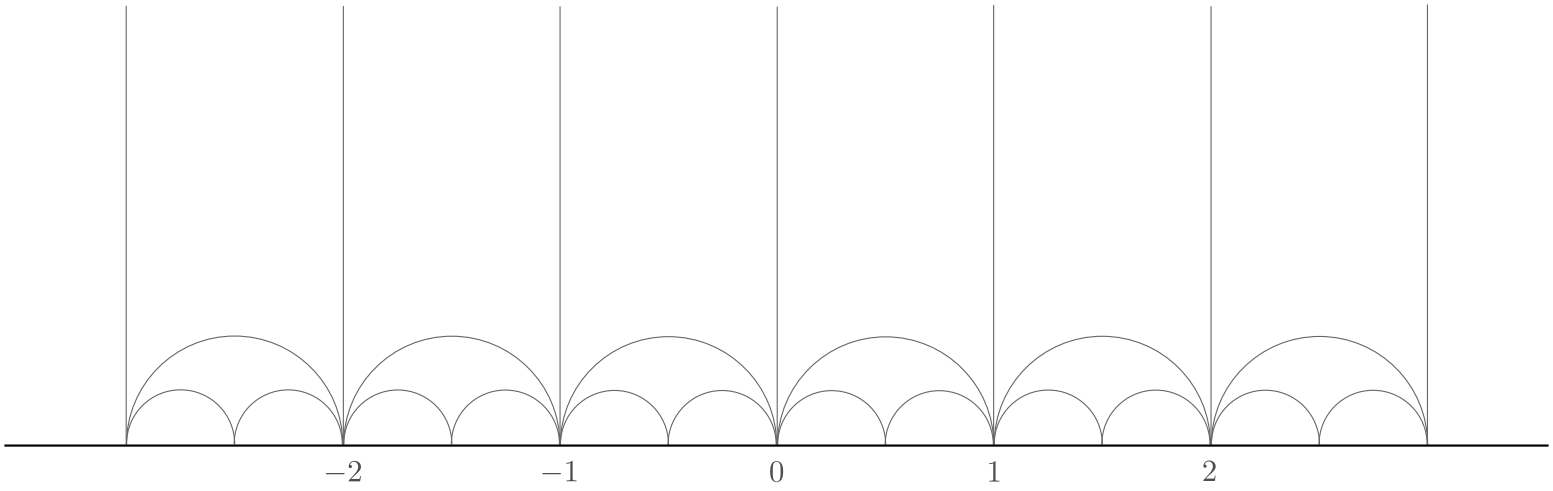
Farey tessellation



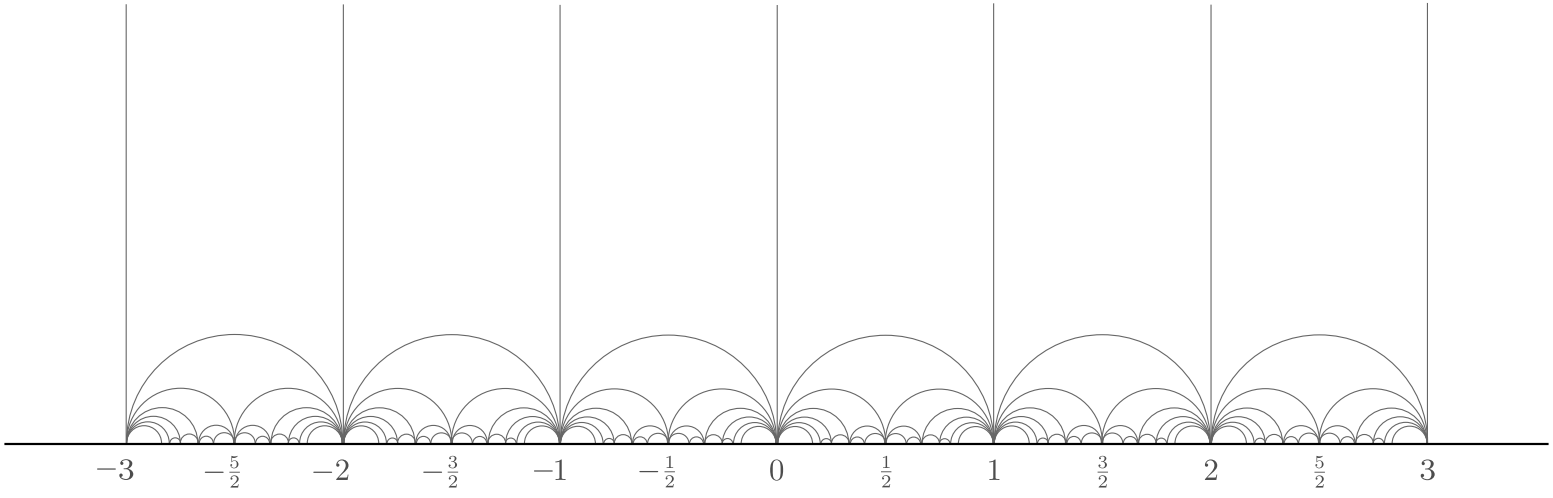
Farey tessellation



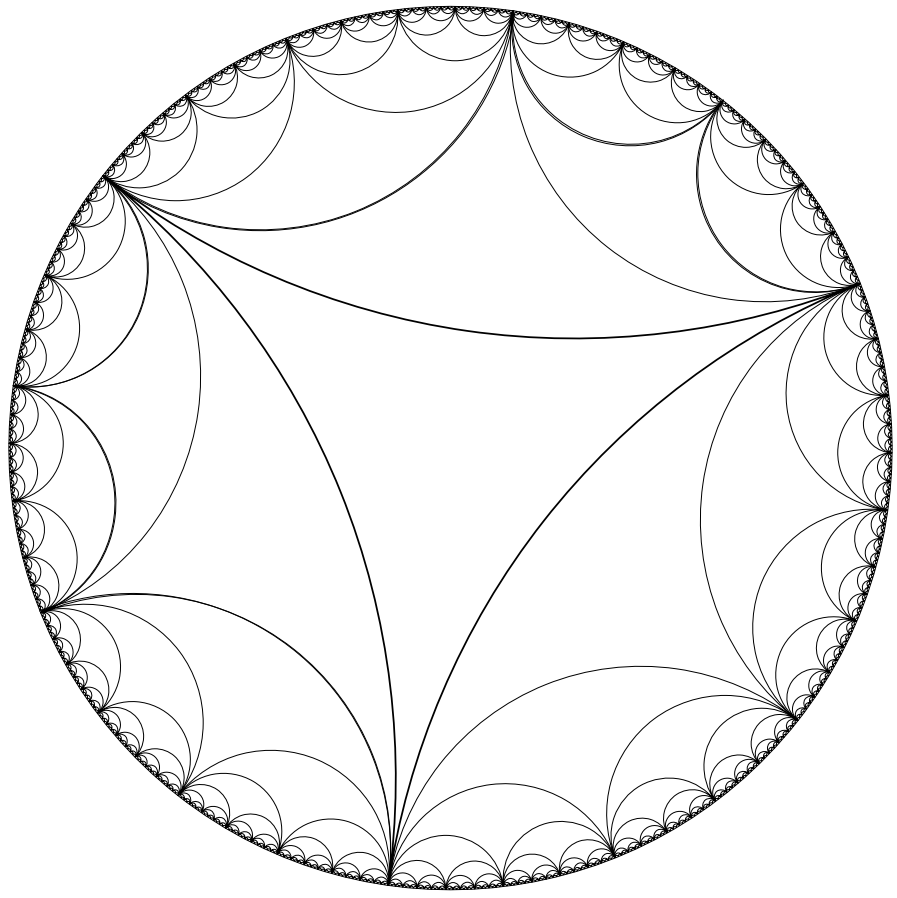
Farey tessellation



Farey tessellation



Farey tessellation in a disc



Embedding triangulated polygons in the Farey tessellation

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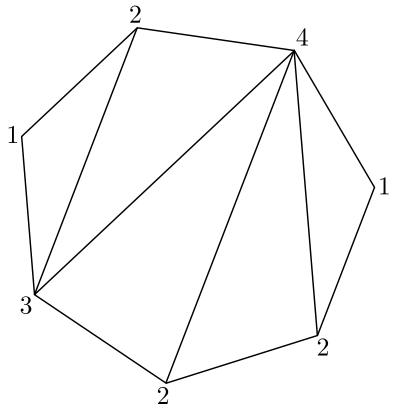

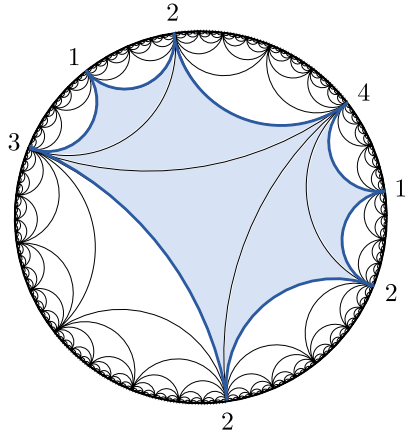

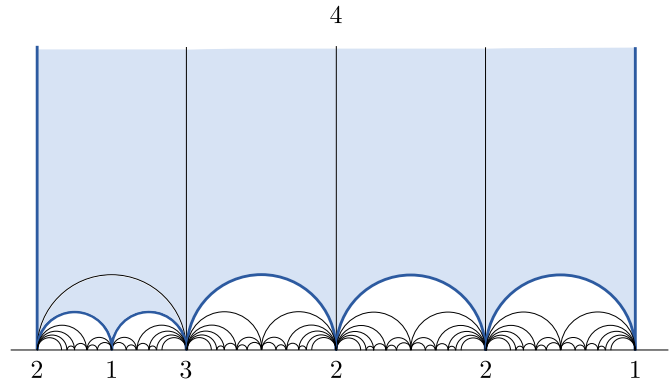

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 & & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & \\
\ldots & & & & & & & & & & & & & & & & & & & & \ldots \\
 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

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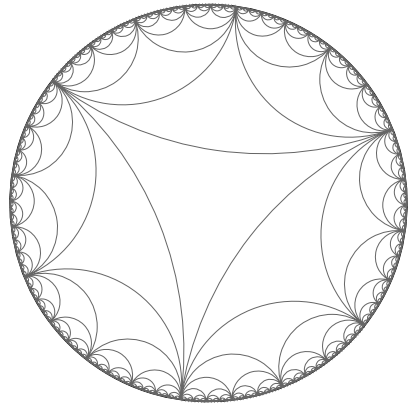
Activity 4

Embedding triangulated polygons in the Farey tessellation



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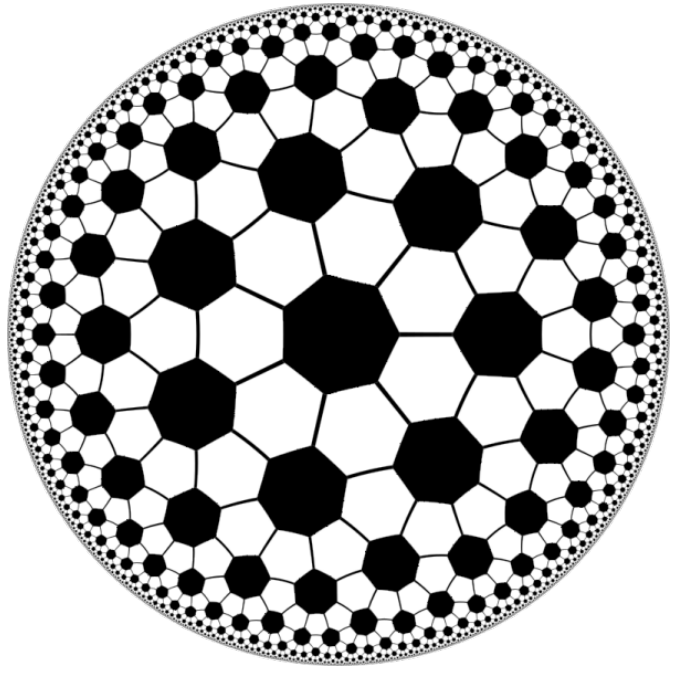
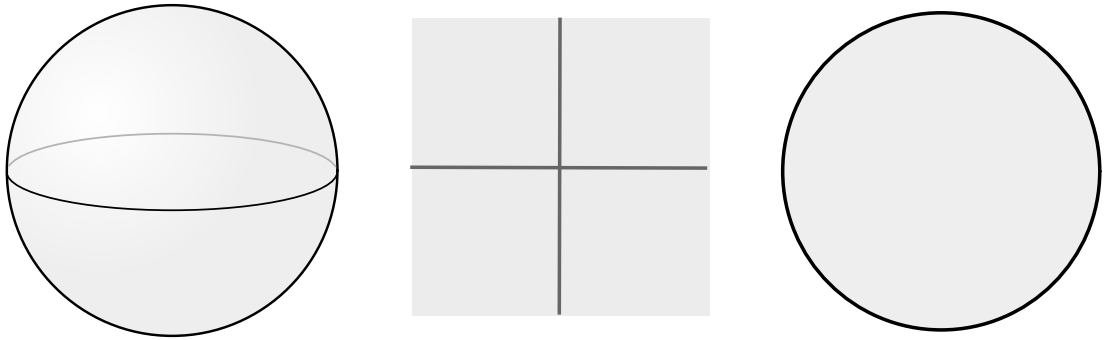
A brief detour around the hyperbolic plane



M.C. Escher's artwork

Circle limit III (1959)

Comparison of geometries



Comparison of geometries

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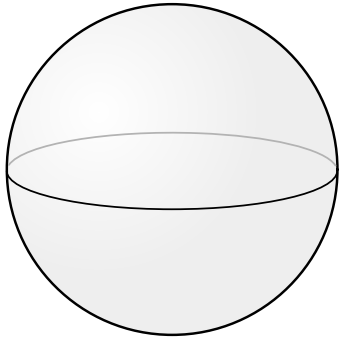
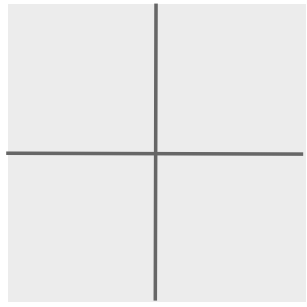
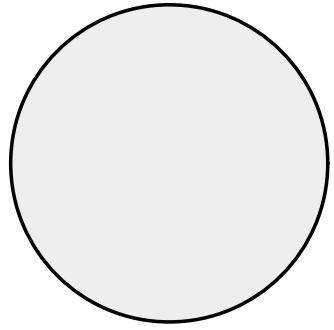


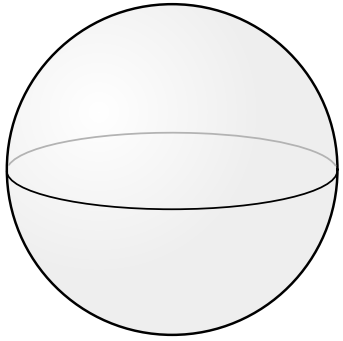


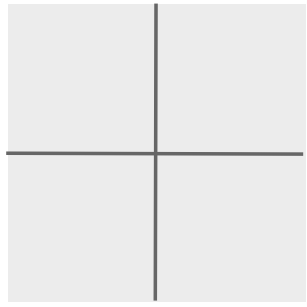


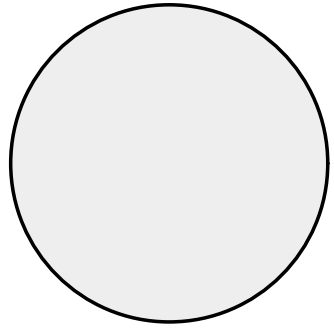
curvature	$+$	0	$-$

Comparison of geometries

The Open University



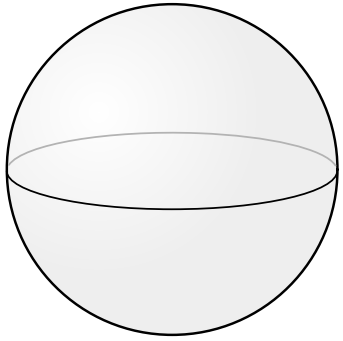


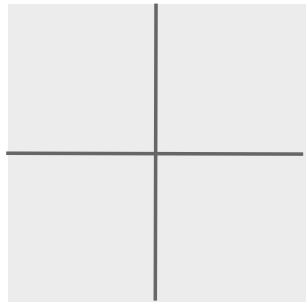


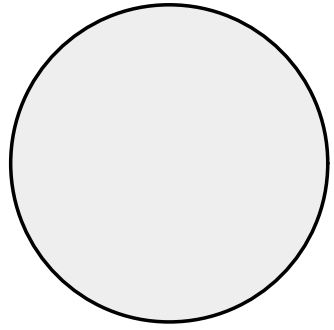
curvature	$+$	0	$-$
triangle angles	$> 180°$	$180°$	$< 180°$

Comparison of geometries

The Open University

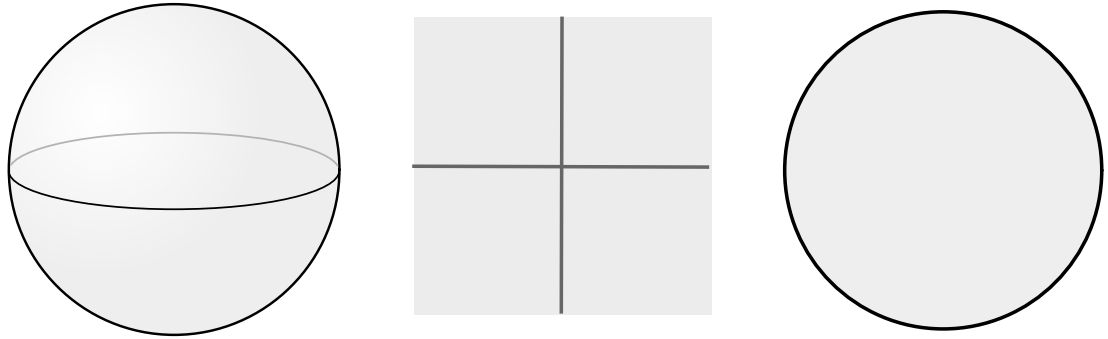






curvature	$+$	0	$-$
triangle angles	$> 180°$	$180°$	$< 180°$
circumference	$< 2\pi r$	$2\pi r$	$> 2\pi r$

Comparison of geometries



curvature	$+$	0	$-$
triangle angles	$> 180°$	$180°$	$< 180°$
circumference	$< 2\pi r$	$2\pi r$	$> 2\pi r$

Example

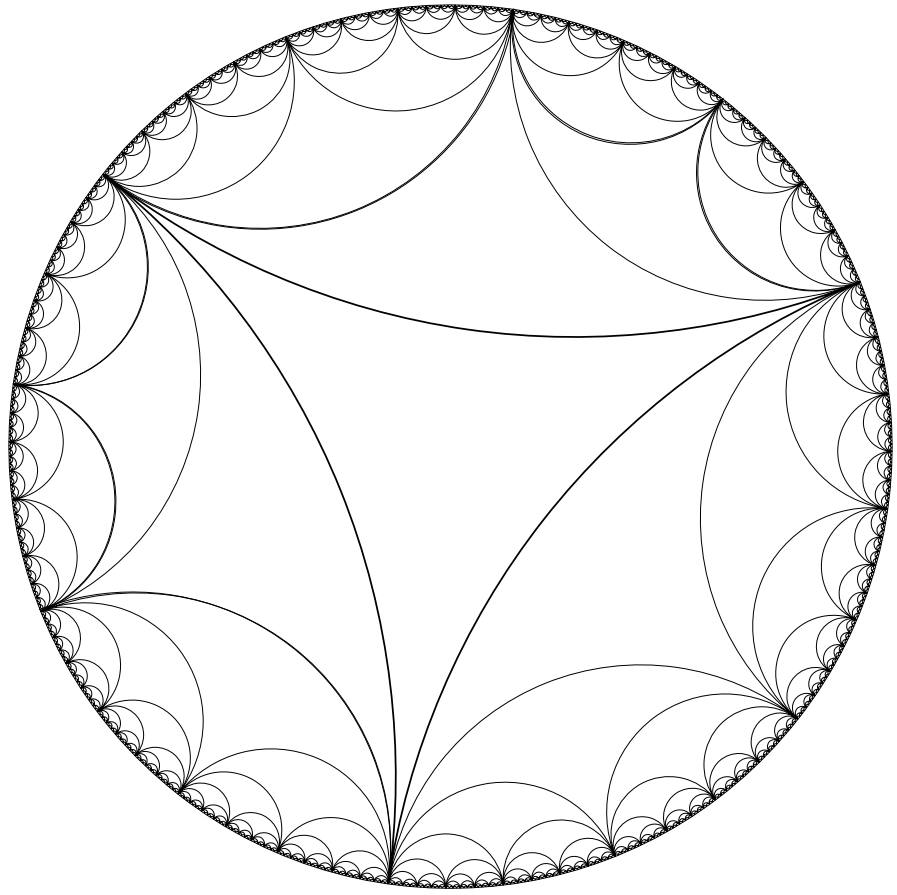
A Euclidean disc of radius 10 has circumference $\simeq 60$.

A hyperbolic disc of radius 10 has circumference $\simeq 70\ 000$.

Beltrami's paper models

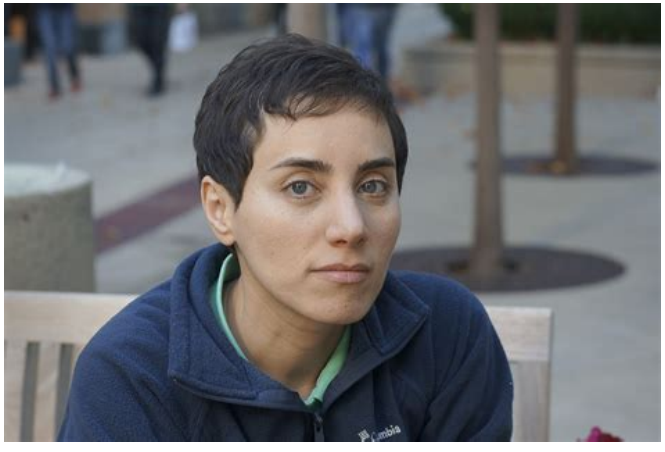
Paper model of the hyperbolic plane by Eugenio Beltrami (1835–1900)

Farey tessellation



The Farey tessellation comprises congruent hyperbolic triangles with angles of zero degrees.

Hyperbolic geometry research thriving



Maryam Mirzakhani (1977–2017)

Awarded the Fields Medal in 2014 for *outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces*.

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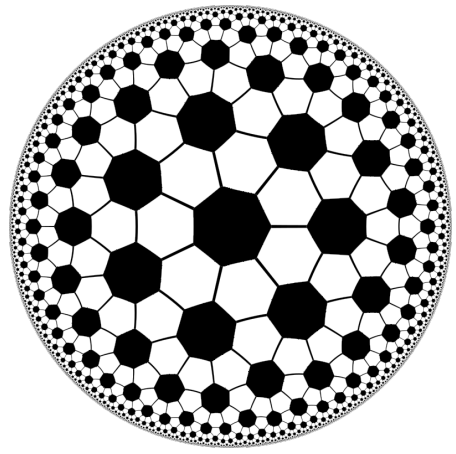
Activity 5

Paper models of the hyperbolic plane



http://www.theiff.org/images/IFF_HypSoccerBall.pdf