MS327

Diagnostic Quiz

Are you ready for MS327?

This diagnostic quiz is designed to help you decide if you are ready to study *Deterministic and stochastic dynamics* (MS327). This document also contains some advice on preparatory work that you may find useful before starting MS327 (see below and page 7). The better prepared you are for MS327 the more time you will have to enjoy the mathematics, and the greater your chance of success.

The topics which are included in this quiz are those that we expect you to be familiar with before you start the module. If you have previously studied *Mathematical methods, models and modelling* (MST210) or *Mathematical methods* (MST224) then you should be familiar with most of the topics covered in the quiz.

We suggest that you try this quiz first without looking at any books, and only look at a book when you are stuck. You will not necessarily remember everything; for instance, for the calculus questions you may need to look up a set of rules for differentiation or integration or use the table of standard derivatives or integrals provided on page 2. This is perfectly all right, as such tables are provided in the Handbook for MS327. You need to check that you are able to use them though.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for MS327. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 7.

Try the questions now, and then see the notes on page 7 of this document to see if you are ready for MS327. (The answers to the questions begin on page 8.)

Do contact your Student Support Team via StudentHome if you have any queries about MS327, or your readiness to study it.

1.4

Tables of standard derivatives and integrals

Function	Derivative
a	0
x^a	ax^{a-1}
e^{ax}	ae^{ax}
$\ln(ax)$	$\dfrac{1}{x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\mathrm{cosec}^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\mathrm{cosec}(ax)$	$-a\mathrm{cosec}(ax)\cot(ax)$
$\arcsin(ax)$	$\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\dfrac{a}{1+a^2x^2}$
$\mathrm{arccot}(ax)$	$-\dfrac{a}{1+a^2x^2}$
$\mathrm{arcsec}(ax)$	$\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$
$\mathrm{arccosec}(ax)$	$-\dfrac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$

Function	Integral
a	ax
$x^a\ (a \neq -1)$	$\dfrac{x^{a+1}}{a+1}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
e^{ax}	$\dfrac{1}{a}e^{ax}$
$\ln(ax)$	$x(\ln(ax)-1)$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax)$
$\cos(ax)$	$\dfrac{1}{a}\sin(ax)$
$\tan(ax)$	$-\dfrac{1}{a}\ln\lvert\cos(ax)\rvert$
$\cot(ax)$	$\dfrac{1}{a}\ln\lvert\sin(ax)\rvert$
$\sec(ax)$	$\dfrac{1}{a}\ln\lvert\sec(ax)+\tan(ax)\rvert$
$\mathrm{cosec}(ax)$	$\dfrac{1}{a}\ln\lvert\mathrm{cosec}(ax)-\cot(ax)\rvert$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)$
$\mathrm{cosec}^2(ax)$	$-\dfrac{1}{a}\cot(ax)$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{(x-a)(x-b)}$	$\dfrac{1}{a-b}\ln\left\lvert\dfrac{a-x}{x-b}\right\rvert$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\ln(x+\sqrt{x^2+a^2})\quad \mathit{or}\quad \mathrm{arcsinh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\ln\lvert x+\sqrt{x^2-a^2}\rvert\quad \mathit{or}\quad \mathrm{arccosh}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right)$

Diagnostic quiz questions

1 Differentiation

Question 1

Differentiate the following functions with respect to x.

(a) $f(x) = \dfrac{x^2 + 2x - 1}{3x + 4}$ (b) $f(x) = \sin(3x^2 + 2)$

Question 2

Find the second derivative with respect to x of $f(x) = 3x^4 + 2\sin 3x$.

Question 3

Find and classify the stationary points of $y = x^3 + 3x^2 - 24x + 10$.

2 Integration

Question 4

Calculate the following integrals.

(a) $\displaystyle\int_1^2 \left(x^2 + \frac{2}{x}\right) \mathrm{d}x$ (b) $\displaystyle\int x^2(x^3 + 5)^{10}\, \mathrm{d}x$ (c) $\displaystyle\int x \cos 3x \, \mathrm{d}x$

3 Ordinary differential equations

Question 5

Find the particular solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{xy}{x^2 + 1}$, where $y > 0$, which satisfies $y(1) = 5\sqrt{2}$.

Question 6

Find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2y}{x} = x$.

Question 7

Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 5x + 8$$

that satisfies $y(0) = 4$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}(0) = 9$.

4 Partial differentiation

Question 8

Find the second-order partial derivatives of the function
$u(x, y) = \cos(4x + 3y)$,

and confirm that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$.

Question 9

Use the Chain Rule to find $\dfrac{\mathrm{d}u}{\mathrm{d}t}$, where $u(x, y) = \ln\left(x^2 y^4\right)$ and
$x = \cos 2t,\ \ y = \sin t$.

5 Matrices

Question 10

Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix}$.

Calculate each of the following, or state why the calculation is not possible.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $2\mathbf{A} + 3\mathbf{B}$ (d) $2\mathbf{B} + \mathbf{C}$

Question 11

Calculate the determinants of the following matrices.

(a) $\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 1 & 0 \\ 3 & 4 & 2 \\ 1 & 4 & 2 \end{pmatrix}$

Question 12

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 8 \\ -1 & -5 \end{pmatrix}$.

Question 13

Find the eigenvalue of the matrix $\begin{pmatrix} 26 & 1 & -5 \\ 4 & 20 & -4 \\ 4 & 8 & 8 \end{pmatrix}$ that corresponds to the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

6 Vector algebra and calculus

Question 14

If $\mathbf{a} = 3\,\mathbf{i} + 2\,\mathbf{j} - 5\,\mathbf{k}$ and $\mathbf{b} = 2\,\mathbf{i} - \mathbf{j} - \mathbf{k}$, find

(a) $\mathbf{a} + \mathbf{b}$, $\quad |\mathbf{a} + \mathbf{b}|$, $\quad \mathbf{a} \cdot \mathbf{b}$, $\quad$ and $\quad \mathbf{a} \times \mathbf{b}$,

(b) the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree.

Question 15

If $\phi(x, y, z) = x^2 y + 2xz$, find $\boldsymbol{\nabla}\phi$ (also known as $\mathbf{grad}\,\phi$) at the point $(2, -2, 3)$.

Question 16

If $\mathbf{F} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find $\boldsymbol{\nabla} \cdot \mathbf{F}$ (also known as $\operatorname{div}\mathbf{F}$) and $\boldsymbol{\nabla} \times \mathbf{F}$ (also known as $\mathbf{curl}\,\mathbf{F}$) at the point $(1, 2, -1)$.

7 Multiple integration

Question 17

Evaluate the surface integral $\displaystyle\int_S xy\,\mathrm{d}A$, where S is the region of the plane $z = 0$ bounded by the curve $y = x^2$ and by the line $y = x$.

Question 18

Evaluate the volume integral $\displaystyle\int_B \left(x^2 y + 2xz^2\right)\,\mathrm{d}V$, where B is the interior of the cube with faces in the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$ and $z = 1$.

8 Mechanics

Question 19

The acceleration of particle of mass 5 kg is given by $\mathbf{a} = 3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k}$.

At $t = 0$ the velocity, $\mathbf{v}(0) = \mathbf{i}$ and the position, $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$.

(a) Find the magnitude of the force acting on the particle.

(b) Find the position of the particle at any time t.

Question 20

A particle connected to a spring moves in a straight line. The position of the particle along the line, x, is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 0.$$

Initially, $x(0) = 3$ and $x'(0) = 0$.

(a) What is the speed of the particle as it passes the point $x = 0$ for the first time?

(b) What is the period of the motion?

9 Complex numbers

Question 21

If $w = 2 + i$ and $z = 3 - 4i$ find, in Cartesian form

(a) wz (b) w/z

Question 22

Express $w = 1 + i$ and $z = \sqrt{3} - i$ in complex exponential form ($re^{i\theta}$), and hence express the following also in complex exponential form.

(a) wz (b) w/z

10 Fourier series

Question 23

Find a Fourier series for the function $f(x) = 1 + x$ for $-1 < x < 1$.

What can I do to prepare for MS327?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what, if anything, you should do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before.	You should consider taking MST210 or MST224 before MS327. Contact your Student Support Team to discuss this.
Some of these topics are familiar to me, but not all of them.	If you have met less than 75% of these topics then you should consider taking MST210 or MST224 before MS327. Contact your Student Support Team to discuss this. If you have taken MST224, then the material in Section 8 might be new to you. If so, you might like to review some of the resources suggested below.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them.	You will benefit from doing some revision before starting MS327 – see below for possible sources of help.
I can do all these questions.	You are probably well prepared for MS327, but do brush up on any topics you need to further enhance your confidence.

If you have any queries contact your Student Support Team via StudentHome.

What resources are there to help me prepare for MS327?

If you have studied MST210, its predecessor MST209, or MST224, then you could use parts of these to revise for MS327.

If you need to brush up some of the more basic topics like algebra, trigonometry and calculus, then you may find revising material from *Essential mathematics 1* (MST124) and *Essential mathematics 2* (MST125) (or their predecessors MST121 and MS221) helpful. Alternatively, such material is often covered by standard A-level textbooks.

If you have studied MST224 rather than MST210 then you might not have met mechanics before (Section 8 of the diagnostic quiz). You might like to review some material on this, such as the MST210 Bridging material available at `mcs-notes2.open.ac.uk/WebResources/Maths.nsf/A/odm8/$FILE/mst210bridging.pdf`, or an A-level mechanics textbook.

The mathcentre web-site (`www.mathcentre.ac.uk`) includes several teach-yourself books, summary sheets, revision booklets, online exercises and video tutorials on a range of mathematical skills.

Diagnostic quiz solutions

Solution 1

(a) $f'(x) = \dfrac{(3x+4)(2x+2) - 3(x^2+2x-1)}{(3x+4)^2} = \dfrac{3x^2+8x+11}{(3x+4)^2}$

(b) $f'(x) = 6x\cos(3x^2+2)$

Solution 2

$f'(x) = 12x^3 + 6\cos 3x, \quad$ so, $f''(x) = 36x^2 - 18\sin 3x$.

Solution 3

The stationary points occur when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, that is, when $3x^2 + 6x - 24 = 0$.

This can be written $3(x-2)(x+4) = 0$, so the stationary points are $x = 2$ and $x = -4$.

At $x = 2$, $y = -18$ and at $x = -4$, $y = 90$.

The second derivative, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 6$.

At $x = 2$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, so $(2, -18)$ is a minimum.

At $x = -4$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, so $(-4, 90)$ is a maximum.

Solution 4

(a) $\displaystyle\int_1^2 \left(x^2 + \frac{2}{x}\right) \mathrm{d}x = \left[\frac{1}{3}x^3 + 2\ln x\right]_1^2 = \frac{8}{3} + 2\ln 2 - \frac{1}{3} = \frac{7}{3} + 2\ln 2.$

(b) $\displaystyle\int x^2(x^3+5)^{10}\,\mathrm{d}x = \frac{1}{33}(x^3+5)^{11} + c$, where c is an arbitrary constant.

(Using the substitution $u = x^3 + 5$.)

(c) Integrating by parts gives

$$\begin{aligned}\int x\cos 3x\,\mathrm{d}x &= x\frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x\,\mathrm{d}x \\ &= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c\end{aligned}$$

where c is an arbitrary constant.

Solution 5

The differential equation is of first order, and the right-hand side can be written as $f(x)g(y)$, for some functions $f(x)$ and $g(y)$. So the equation can be solved using separation of variables.

Rearrange the equation so that all the terms involving y are on the left-hand side, and all those involving x are on the right:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{x^2+1}.$$

Integrating both sides gives $\displaystyle\int \frac{1}{y}\,\mathrm{d}y = \int \frac{x}{x^2+1}\,\mathrm{d}x$

hence, $\displaystyle\ln y = \frac{1}{2}\ln(x^2+1) + c,$ where c is an arbitrary constant.

So $y = \exp\left(\frac{1}{2}\ln(x^2+1)+c\right) = A\sqrt{x^2+1}$,where $A = e^c$.

Since $y = 5\sqrt{2}$ when $x = 1$, we have $5\sqrt{2} = A\sqrt{2}$, so $A = 5$ and hence $y = 5\sqrt{x^2+1}$.

Solution 6

The equation is of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = f(x)$ so can be solved using the integrating factor method. The integrating factor is $p(x) = \exp\left(\int g(x)\,\mathrm{d}x\right)$, which in this case is

$$p(x) = \exp\left(\int \frac{2}{x}\,\mathrm{d}x\right) = \exp\left(2\ln x\right) = x^2.$$

So, $\displaystyle x^2y = \int \left(x^2 \times x\right)\,\mathrm{d}x = \int x^3\,\mathrm{d}x = \frac{x^4}{4} + c$, where c is an arbitrary constant.

Hence $\displaystyle y = \frac{x^2}{4} + \frac{c}{x^2}.$

Solution 7

First solve the associated homogeneous equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0,$

The solutions are, $\lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i.$

Hence the complementary function is $y_{\mathrm{c}} = e^x(A\cos(2x) + B\sin(2x)),$ where A and B are arbitrary constants.

Returning to the inhomogeneous equation, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 5y = 5x + 8,$

the form of the right-hand side suggests a trial solution of the form $y_{\mathrm{p}} = ax + b.$

So, $\dfrac{\mathrm{d}y_{\mathrm{p}}}{\mathrm{d}x} = a$ and $\dfrac{\mathrm{d}^2y_{\mathrm{p}}}{\mathrm{d}x^2} = 0.$

Substituting into the inhomogeneous equation gives $\quad 0 - 2a + 5(ax + b) = 5x + 8$
which can be written $\quad 5ax + (5b - 2a) = 5x + 8$.

Comparing coefficients gives $\quad 5a = 5 \quad$ and $\quad 5b - 2a = 8$.

So, $\quad a = 1 \quad$ and $\quad b = 2$.

The particular integral is $\quad y_{\mathrm{p}} = x + 2$.

The general solution is $y = y_{\mathrm{c}} + y_{\mathrm{p}}$, that is,
$y = e^x(A\cos(2x) + B\sin(2x)) + x + 2$.

When $x = 0$, $y = 4$, so $4 = A + 2$, and $A = 2$.

The derivative of the general solution is (using the Product Rule)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x((A + 2B)\cos(2x) + (B - 2A)\sin(2x)) + 1.$$

When $x = 0$, $\mathrm{d}y/\mathrm{d}x = 9$, so $\quad 9 = A + 2B + 1 = 2B + 3 \quad$ and so $\quad B = 3$.
The particular solution required is $\quad y = e^x\,(2\cos(2x) + 3\sin(2x)) + x + 2$.

Solution 8

$$\frac{\partial u}{\partial x} = -\sin(4x + 3y) \times 4 = -4\sin(4x + 3y)$$

$$\frac{\partial u}{\partial y} = -\sin(4x + 3y) \times 3 = -3\sin(4x + 3y)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(-4\sin(4x + 3y)) = -4\cos(4x + 3y) \times 4 = -16\cos(4x + 3y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(-3\sin(4x + 3y)) = -3\cos(4x + 3y) \times 4 = -12\cos(4x + 3y)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-3\sin(4x + 3y)) = -3\cos(4x + 3y) \times 3 = -9\cos(4x + 3y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(-4\sin(4x + 3y)) = -4\cos(4x + 3y) \times 3 = -12\cos(4x + 3y)$$

Comparison of $\dfrac{\partial^2 u}{\partial y\,\partial x}$ with $\dfrac{\partial^2 u}{\partial x\,\partial y}$ confirms that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

Solution 9

The Chain Rule gives $\quad \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{\partial u}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{\partial u}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

We know, $\quad \dfrac{\partial u}{\partial x} = \dfrac{1}{x^2} \times 2x = \dfrac{2}{x}, \qquad \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\sin 2t, \qquad \dfrac{\partial u}{\partial y} = \dfrac{1}{y^4} \times 4y^3 = \dfrac{4}{y}, \qquad \dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos t$.

So, $\quad \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{2}{x} \times (-2\sin 2t) + \dfrac{4}{y} \times \cos t = \dfrac{4\cos t}{y} - \dfrac{4\sin 2t}{x}$.

Now substitute for x and y in terms of t: $\quad \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{4\cos t}{\sin t} - \dfrac{4\sin 2t}{\cos 2t} = 4\cot t - 4\tan 2t$.

Solution 10

(a) $\mathbf{AB} = \begin{pmatrix} 31 & 13 \\ 6 & 2 \\ 17 & 7 \end{pmatrix}$

(b) $\mathbf{BA}$ cannot be calculated, since $\mathbf{B}$ is 2×2 and $\mathbf{A}$ is 3×2.

(c) $2\mathbf{A} + 3\mathbf{B}$ cannot be calculated, since $\mathbf{A}$ is 3×2 and $\mathbf{B}$ is 2×2.

(d) $2\mathbf{B} + \mathbf{C} = \begin{pmatrix} 7 & 7 \\ 10 & 4 \end{pmatrix}$

Solution 11

(a) $\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = 5 \times 4 - 2 \times 3 = 14$

(b) $\begin{vmatrix} 4 & 1 & 0 \\ 3 & 4 & 2 \\ 1 & 5 & 1 \end{vmatrix} = 4 \begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 4(4 - 10) - 1(3 - 2) = -25$

Solution 12

The characteristic equation is given by $\begin{vmatrix} 1-\lambda & 8 \\ -1 & -5-\lambda \end{vmatrix} = 0,$

that is $(1-\lambda)(-5-\lambda) + 8 = 0$ or $\lambda^2 + 4\lambda + 3 = 0.$

This can be factorised as $(\lambda + 1)(\lambda + 3) = 0$, hence the eigenvalues are -1 and -3.

The eigenvectors $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ associated with the eigenvalue λ are found by solving

$$\begin{pmatrix} 1-\lambda & 8 \\ -1 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

For $\lambda = -1$ this gives $\begin{pmatrix} 2 & 8 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ which can be written as

$$\begin{aligned} 2x + 8y &= 0 \\ -x - 4y &= 0. \end{aligned}$$

These are satisfied when $x = -4y$, so an eigenvector is $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

For $\lambda = -3$ the equation gives $\begin{pmatrix} 4 & 8 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ which can be written as

$$\begin{aligned} 4x + 8y &= 0 \\ -x - 2y &= 0. \end{aligned}$$

These are satisfied when $x = -2y$, so an eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Solution 13

Multiplying the matrix by the given eigenvector gives

$$\begin{pmatrix} 26 & 1 & -5 \\ 4 & 20 & -4 \\ 4 & 8 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 36 \end{pmatrix} = 12 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

The product is 12 times the eigenvector, so the corresponding eigenvalue is 12.

Solution 14

(a) $\mathbf{a}+\mathbf{b} = (3\,\mathbf{i}+2\,\mathbf{j}-5\,\mathbf{k}) + (2\,\mathbf{i}-\mathbf{j}-\mathbf{k}) = 5\,\mathbf{i}+\mathbf{j}-6\,\mathbf{k}$.

$|\mathbf{a}+\mathbf{b}| = |5\,\mathbf{i}+\mathbf{j}-6\,\mathbf{k}| = \sqrt{5^2+1^2+(-6)^2} = \sqrt{25+1+36} = \sqrt{62}$.

$\mathbf{a}\cdot\mathbf{b} = 3\times 2 + 2\times(-1) + (-5)\times(-1) = 6-2+5 = 9$.

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -5 \\ 2 & -1 & -1 \end{vmatrix} \\ &= (2\times(-1)-(-5)\times(-1))\,\mathbf{i} - (3\times(-1)-(-5)\times 2)\,\mathbf{j} + (3\times(-1)-2\times 2)\,\mathbf{k} \\ &= -7\,\mathbf{i}-7\,\mathbf{j}-7\,\mathbf{k} = -7(\mathbf{i}+\mathbf{j}+\mathbf{k}). \end{aligned}$$

(b) The cosine of the angle θ between $\mathbf{a}$ and $\mathbf{b}$ can be calculated using $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

$|\mathbf{a}| = \sqrt{3^2+2^2+(-5)^2} = \sqrt{38}$ and $|\mathbf{b}| = \sqrt{2^2+(-1)^2+(-1)^2} = \sqrt{6}$.

So, $\cos\theta = \dfrac{9}{\sqrt{38}\sqrt{6}} = \dfrac{3}{2}\sqrt{\dfrac{3}{19}}$.

Hence, $\theta = \arccos\left(\dfrac{3}{2}\sqrt{\dfrac{3}{19}}\right) = 53^\circ$ (to the nearest degree).

Solution 15

The scalar field is $\phi(x,y,z) = x^2y + 2xz$.

The partial derivatives are

$$\frac{\partial\phi}{\partial x} = 2xy+2z, \qquad \frac{\partial\phi}{\partial y} = x^2, \qquad \text{and} \qquad \frac{\partial\phi}{\partial z} = 2x.$$

So, $\quad \boldsymbol{\nabla}\phi = \mathbf{grad}\,\phi = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k} = (2xy+2z)\,\mathbf{i} + x^2\,\mathbf{j} + 2x\,\mathbf{k}$.

At the point $(2,-2,3)$,

$\boldsymbol{\nabla}\phi = \mathbf{grad}\,\phi = (2\times 2\times(-2)+2\times 3)\,\mathbf{i} + 2^2\,\mathbf{j} + 2\times 2\,\mathbf{k} = -2\,\mathbf{i}+4\,\mathbf{j}+4\,\mathbf{k}$.

Solution 16

If $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, then $\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

and $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Here, $\mathbf{F} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, so, $\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = 2xy - 0 + 2y = 2y(x+1)$

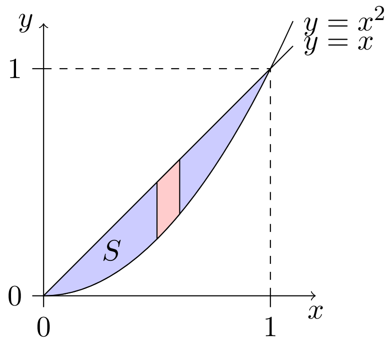
and $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}$

$$= (2z + 2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k} = 2(x+z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}.$$

At $(1, 2, -1)$, $\boldsymbol{\nabla}\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = 2\times 2(1+1) = 8$ and $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{curl}\,\mathbf{F} = 2(1-1)\,\mathbf{i} - (1-2)\,\mathbf{k} = \mathbf{k}$.

Solution 17

First sketch the region of integration, S:



Note that $y = x$ and $y = x^2$ intersect where $x = 0$ and $x = 1$, giving the minimum and maximum values of x for the region S.

In Cartesian coordinates we can write $\displaystyle\int_S xy\,\mathrm{d}A = \int\!\!\int_S xy\,\mathrm{d}y\,\mathrm{d}x,$

in which we integrate with respect to y first, and then respect to x, and hence divide the region of integration into strips as shown in the diagram above.

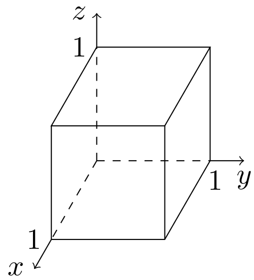
Using the diagram to identify the limits of x and y we obtain $\displaystyle\int_S xy\,\mathrm{d}A = \int_{x=0}^{x=1}\left(\int_{y=x^2}^{y=x} xy\,\mathrm{d}y\right)\mathrm{d}x.$

Now, $\displaystyle\int_{y=x^2}^{y=x} xy\,\mathrm{d}y = x\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x} = x\left(\frac{1}{2}x^2 - \frac{1}{2}x^4\right) = \frac{1}{2}(x^3 - x^5),$

and hence $\displaystyle\int_S xy\,\mathrm{d}A = \int_{x=0}^{x=1} \tfrac{1}{2}(x^3 - x^5)\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}.$

Solution 18

First sketch the region of integration, B:



In Cartesian coordinates, $\displaystyle\int_B \left(x^2y + 2xz^2\right)\,\mathrm{d}V = \iiint_B \left(x^2y + 2xz^2\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Since the boundaries of the volume of integration are coordinate planes, we have

$$\begin{aligned}
\int_B \left(x^2y + 2xz^2)\right)\mathrm{d}V &= \int_0^1\int_0^1\int_0^1 \left(x^2y + 2xz^2\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1\int_0^1 \left[\frac{1}{3}x^3y + x^2z^2\right]_{x=0}^{x=1}\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1\int_0^1 \left(\frac{1}{3}y + z^2\right)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \left[\frac{1}{6}y^2 + z^2y\right]_{y=0}^{y=1}\,\mathrm{d}z \\
&= \int_0^1 \left(\frac{1}{6} + z^2\right)\,\mathrm{d}z \\
&= \left[\frac{1}{6}z + \frac{1}{3}z^3\right]_{z=0}^{z=1}\,\mathrm{d}z = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.
\end{aligned}$$

Solution 19

(a) The force, $\mathbf{F}$, is given by Newton's second law: $\mathbf{F} = m\mathbf{a}$, where m is the mass of the particle.

So, $\mathbf{F} = 5(3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k}) = 15\sin 6t\,\mathbf{i} + 15\cos 6t\,\mathbf{j} + 20\,\mathbf{k}$..

Hence, the magnitude of the force,

$$|\mathbf{F}| = \sqrt{225\sin^2 6t + 225\cos^2 6t + 400} = \sqrt{225 + 400} = \sqrt{625} = 25.$$

(b) The velocity, $\mathbf{v}$, is given by

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int (3\sin 6t\,\mathbf{i} + 3\cos 6t\,\mathbf{j} + 4\,\mathbf{k})\ \mathrm{d}t = -\frac{1}{2}\cos 6t\,\mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k} + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary vector constant.

Since $\mathbf{v}(0) = \mathbf{i}$, we have $\quad -\dfrac{1}{2}\mathbf{i} + \mathbf{c} = \mathbf{i}, \quad$ so $\mathbf{c} = \dfrac{3}{2}\mathbf{i}$ and

$$\mathbf{v} = \left(\frac{3}{2} - \frac{1}{2}\cos 6t\right)\ \mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k}.$$

The position, $\mathbf{r}$, is given by

$$\begin{aligned}\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t &= \int \left(\left(\frac{3}{2} - \frac{1}{2}\cos 6t\right)\ \mathbf{i} + \frac{1}{2}\sin 6t\,\mathbf{j} + 4t\,\mathbf{k}\right)\ \mathrm{d}t \\ &= \left(\frac{3}{2}t - \frac{1}{12}\sin 6t\right)\ \mathbf{i} - \frac{1}{12}\cos 6t\,\mathbf{j} + 2t^2\,\mathbf{k} + \mathbf{d},\end{aligned}$$

where $\mathbf{d}$ is an arbitrary vector constant.

Since $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, we have $\quad -\dfrac{1}{12}\mathbf{j} + \mathbf{d} = \mathbf{j} + \mathbf{k}, \quad$ so $\mathbf{d} = \dfrac{13}{12}\mathbf{j} + \mathbf{k}$ and

$$\mathbf{r} = \left(\frac{3}{2}t - \frac{1}{12}\sin 6t\right)\ \mathbf{i} + \left(\frac{13}{12} - \frac{1}{12}\cos 6t\right)\ \mathbf{j} + \left(1 + 2t^2\right)\ \mathbf{k}.$$

Solution 20

(a) The solution of the differential equation is $\qquad x = A\cos 2t + B\sin 2t,$

for arbitrary constants A and B.

When $t = 0$, $x = 3$, so $\quad 3 = A$ and $\quad x = 3\cos 2t + B\sin 2t$..

Hence $x'(t) = -6\sin 2t + 2B\cos 2t$, and since $x'(0) = 0$, then $2B = 0$ and so $B = 0$.

So the motion is given by $\qquad x = 3\cos 2t$.

The particle is at $x = 0$ when $3\cos 2t = 0$, that is $\cos 2t = 0$.

The first solution for $t > 0$ is $2t = \pi/2$, that is $t = \pi/4$.

We know $x'(t) = -6\sin 2t$, so at $t = \pi/4$: $\quad x'(t) = -6\sin\frac{\pi}{2} = -6$.

So the speed of the particle as it crosses $x = 0$ for the first time is 6.

(b) The angular frequency, $\omega = \sqrt{4} = 2$. So the period is given by $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$.

Solution 21

(a) $w + 2z \qquad (2+i)(3-4i) = 6 - 8i + 3i - 4i^2 = 6 - 5i + 4 = 10 - 5i.$

(b) $\dfrac{w}{z} = \dfrac{2+i}{3-4i}$. This can be expressed in Cartesian form by rationalising the denominator.

$$\frac{2+i}{3-4i} = \frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{6+8i+3i+4i^2}{9-16i^2} = \frac{2+11i}{25} = \frac{2}{25} + \frac{11}{25}i.$$

Solution 22

Since $w = 1+i$ then $|w| = \sqrt{1^2+1^2} = \sqrt{2}$ and $\arg(w) = \arctan(1/1) = \pi/4$, so w has complex exponential form $w = \sqrt{2}e^{i\pi/4}$. Likewise $z = \sqrt{3} - i$ so $|z| = \sqrt{\sqrt{3}^2 + 1^2} = 2$ and $\arg(z) = \arctan(-1/\sqrt{3}) = -\pi/6$, so z has complex exponential form $z = 2e^{-i\pi/6}$.

(a) $wz = \sqrt{2}e^{i\pi/4} \times 2e^{-i\pi/6} = 2^{3/2}e^{i\pi/12}$

(b) $w/z = \dfrac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \dfrac{1}{\sqrt{2}}e^{5i\pi/12}.$

Solution 23

The Fourier series for a function $f(x)$ over $-L < x < L$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \qquad \text{and} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x.$$

Here, $f(x) = 1 + x$ and $L = 1$, so

$$a_0 = \int_{-1}^{1}(1+x)\,\mathrm{d}x = \left[x + \frac{x^2}{2}\right]_{-1}^{1} = 1 + \frac{1}{2} - \left(-1 + \frac{1}{2}\right) = 2$$

$$\begin{aligned} a_n = \int_{-1}^{1}(1+x)\cos n\pi x\,\mathrm{d}x &= \left[(1+x)\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{1} - \int_{-1}^{1}\left(1 \times \frac{1}{n\pi}\sin n\pi x\right)\,\mathrm{d}x \\ &= 0 + \frac{1}{n^2\pi^2}\left[\cos n\pi x\right]_{-1}^{1} = \frac{1}{n^2\pi^2}\left(\cos n\pi - \cos(-n\pi)\right) = 0 \end{aligned}$$

$$\begin{aligned} b_n = \int_{-1}^{1}(1+x)\sin n\pi x\,\mathrm{d}x &= \left[-(1+x)\frac{1}{n\pi}\cos n\pi x\right]_{-1}^{1} + \int_{-1}^{1}\left(1 \times \frac{1}{n\pi}\cos n\pi x\right)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_{-1}^{1} \\ &= -\frac{2}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\left(\sin n\pi - \sin(-n\pi)\right) = -\frac{2}{n\pi}\cos n\pi = -\frac{2}{n\pi}(-1)^n \end{aligned}$$

Hence, the Fourier series is $\qquad F(x) = 1 - \displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^n \sin\frac{n\pi x}{L}.$