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L'HOPITAL'S RULE

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Generic Titles

DAVID BRANNAN:

Studio V/O

Suppose you wanted to sketch the curve which has this equation.

In this interval

An obvious first step is to look at the behaviour of two simpler curves.

Here's y = cos 3x

call that f(fx)

and here's y = sin x-e to the cos x
call that g(x)

It's clear that f and g are continuous, at least in this interval. But does this mean that f/g, the quotient function you're looking for, is also continuous?

Well, if theres going to be a problem, it will be here, at x = 11/2 where both the numerator and the denominator are zero.

So, what does the quotient function look like for points ner to 11/2?

To the right of 11/2 both numberator and denominator are positive.

To the left of 11/2 f(x) and g(x) are both negative and so their quotient will be positive there as well. But what happens at x = 11/2 itself?

Since g(x) disappeas there, you might expect that f/g has a vertical asymptote. Well, let'lets plot some values for f/g.

The only point we can't evaluate it at is x = 11/2, of course, but it looks from the two sides of the graph as if they should meet there with the value 3.

D. Brannan (continued)

So x = 11/2 isn't a vertical asymptote and the reason for that is because the numerator vanishes as well as the denominator. But we do seem to have a limit.

Are we able to say whether this limit does exist?

And then if it does, does it equal 3?

ANIMATE 1

David and Board 1

widen to include bottom row

Let me ask those questions more generally. If you have two continuous functions f and g which are both zero at the same point c, what can you say about their ratio as x tends to c? Does it exist? Can you evaluate it?

Well by the end of the programme you'll have a theorem, called L'Hopital's Rule after the 17th Century Mathematician, Marquis Guillaume de L'Hopital - which answers these questions and I think it's an amazing result!

Remove Label

What you do is take the derivatives of numerator and denominator and then consider the limit of their ratio. And under certain conditions which we'll add in later the theorem says that the original limit does exist and it equals this as long as this exists.

Widen and follow walk

Board 2 and

Unmasked boxes.

Well, that's L'Hopital's Rule and if you use it on the function you've already met it does give the value 3. I'll be showing you that later on, along with what happens when you try it on this other function.

Both numerator and denominator vanish at zero this time. But if you're going to use l'Hopital's Rule here you'll find you have to be really rather crafty.

ROY NELSON:

Chyron ident.

Well, there's a lot to be done before you

R. Nelson (continued)

can be satisfied that the Rule is correct.

MCU 4 Quadrants

2/1 to top left Quad.

We are going to go through separate steps in order to prove l'Hopital's Rule. We begin withRolle's Theorem which you've already met in the Unit and although that doesn't seem to have much to do with proving l'Hopital's Rule you'll find that we can build on that to give other theorems that can be used.

CAPTION 1

CU GRAPH

So let me remind you what Rolle said. For a function f, differentiable in the open interval a,b we suppose that f(a)=f(b) Rolle's theorem then says that there is some point k inside a,b for which f'(k)=0:- the tangent to the curve at k is horizontal. In other words, the tangent is parallel to the chord joining the end points of the curve.

Editec on graphic

CU Graph

Our next step is to look at an extension of Rolle's theorem called the MVT to cover the general case where f(a) is not necessarily equal to f(b).

In this case it's not a horizontal tangent we're interested in butwe're still looking for a point for which the tangent to the curve is parallel to this chord.

Roy and Board

In fact, it's easy enough to find such a point k, if we look at the situation geometrically. We start with a line parallel to the chord and then we move it vertically keeping it always parallel to the chord intersecting the curve in two points until eventually it will meet the curve at one point.

Roy and Board 3

It is then a tangent and the point where it touches the curve gives us the point k we seek.

In other words, it is the point of the curve for which this vertical distance is a maximum. So how do we find that maximum? Let's look at it in a different way.

Widen to follow walk.

CU Diagrams

Over here i've reproduced the axes with the same interval. At each point x of the interval, let h(x) denote just the distance above the chord to the curve. Now the point we are looking for is one

R. Nelson (continued)

for which h(x) is a maximum. In other words we are trying to find a point where h dashed is zero. But of course, before we can differentiate, we need to know what the function h(x) is.

2 Roy and perspex

Tilt down

Well, in terms of f(x),we have h(x)=f(x) - this distance (x tan alpha plus 1). This is an elementary result, which is just , this is alpha, this is 1 the intercept on the axis givingus the height up to the chord.

So now we can differentiate to get: h'(x) = f'(x)-tan alpha. Now, remember what we are looking for is a point k for which h'(x) = 0 that means we require f'(k) = tan alpha. In other words the tangent at k has the same slope as the chord.

Roy and perspex

Of course, we haven't proved anything yet! We have not shown that such apoint k actually exists. But we seem to be on the right track: if the point k does exist, then we shall have found a point at which h(x) is a maximum.

So what do we do next?

CU graph.

You see to know that k exists we don't actually need to look for maxima, just to know that h'(x) takes a zero value somewhere. Can we prove it does? Yes, we can because h satisfies Rolle's Conditions continuity, differentiability and also we have h(a) and h(b) are both zero. So h' disappears somewhere inside the interval.

add on $\frac{f(b) - f(a)}{b - a}$)

And since h'(k)=0 therefore f'(k)-tan alpha=0 so f'(k) does = tan alpha. Now this really is what the MVT says. And in fact we can express the chord slope in another way.

Roy R of Board 6

The tangent of this angle is of course the difference in vertical heights divided by this horizontal distance. We've relied on the geometry to prove this result for this particular curve but if you look in the unit you'll find a rigourous proof applying to any suitable curve. Well, that's two theorems dealt with, we're half way to the proof of L'Hopital's Rule, but

R. Nelson (continued)

we still don't seem to be very close to it. It's the next theorem that brings us a lot closer and we are now able to tackle that. We need to look at the parametric plane.

Because if you look at the equivalent theorem for that plane you get a ratio of two functions.

DAVID BRANNAN:

In the parametric plane, curves have their x and y coordinates described by independent function, f(t) and g(t). In other words, the curves resulting from plotting points for different values of t.

So you can think of the curve as representing the path of a particle going along the curve from one point to another.

You can think of t as time.

Let's call the beginning t=a and the other end t = b. It's the same sort of theorem again.

Here's a chord joining the end points. Once again there is going to be at least one point on the curve, where a tangent to the curve is a parallel to this chord.

End Animation 2

Call such a point t=k. What we've got here geometrically is another MVT. The Cauchy MVT gives a version of the MVT for curves given in parametric form.

Diagram

David and Board 4

It says there exists a point k such that the tangent slope equals this expression.

Equation

Walk to Board 5

How would you work out the slope of a parametric tangent? It's the limit of chords through the point t=k in this case - the slope of the chord is this height divided by this base - so you can work out the slope of the tangent as this completed limit.

Bottom Board 5.

It's certainly true that I can divide all the tope expression by h if I do the same to the denominator.

D. Brannan (continued)

Which can of course be expressed as a quotient of two limits and so now the numerator is our familiar definition of the derivative f'(k), and similarly the denominator is g'(k).

So the tangent slope at k is f'(k)/g'(k). This is what's needed to state the Cauchy MVT fully.

I've given you the idea of the proof geometrically. The rigourous proof is in the text as it's too algebraic to do here.

Add $\frac{f\ (\quad)}{g\ (k)}$

but this is the formal result . . . and at least this bit is reminiscent of L'Hopital's Rule.

ROY NELSON:

Bottom Board 1

But how does such a MV theorem for parametric curves get us closer to solving problems of the general type we started the programme with. The condition was that f and g should vanish together at some point, c.

Well, in the parametric plane, f and g vanish at the origin. Call that the point t=c. So we have only to consider the special case of parametric curves which actually pass through the origin.

Any result which we obtain in this case can be used to tackle our original problems.

Diagram and Roy

So whatever our curve is, it must go through the origin, and then on, maybe something like this. Suppose that the beginning of the curve is now here.

So what can we say about it?

A curve passing through the origin at t=c. If you think of t as time, the part of the curve from a to c corresponds to points earlier than c. We continue to follow the curve through points later than c, as far as the point b.

R. Nelson (continued)

Let's look at that part of the curve first and see what we can say about the chord joining the two point c and b.

We can apply Cauch's MVT to find a point t=k in the interval c,b where the tangent is parallel to the chord.

Graph and Roy

We then have $\frac{f'(k)}{g'(k)} = \frac{f(b)-f(c)}{g(b)-g(c)}$ just as before.

But now of course, we are subtracting f(c)andg(c)-both of which are zero. So what we get is

$$\frac{f'(k)}{g'(k)} = \frac{f(b)}{g(b)}$$

Now imagine that b moves closer and closer to c. Then the chord itself will move closer to the tangent at the origin and also as b decreases to, the point k is pushed closer and closer to c. In fact, b and k tend to c+, together.

MCU Panel

This equality always holds throughout the limiting process as k and b tends to c+. Now if we suppose that this limit

$$\frac{f'(k)}{k-c+g'(k)}$$

exists, it must follow that limit

$$\frac{f(b)}{bc+g(b)}$$

also exists and in fact they'll have the same value.

CU Origin

Notice that b and k have now assumed the same role, that of a variable approaching c from the right. So we might as well replace each of them by the symbol 't'.This limit is then the limit $\frac{f'(t)}{t-c+g'(t)}$

CU hand and panel

and this limit becomes the limit

$$\frac{f(t)}{t-c+g(t)}$$

Widen include

top box

In fact, if this limit as t-c is known to exist then both these two one sided limits are known to exist and they must be equal to it.

Bottom left Quad.

Has this proved L'Hopital's Rule? It looks very promising. But don't forget these are all 'right' limits. Well, you

R. Nelson (continued)

use exactly the same argument to get the corresponding result with 'left' limits.

Once again if we know that $\lim_{t \to c} \frac{f'(t)}{g'(t)}$ exists then these left hand limits exist and are equal to it.

So what does that give us? Let's bring the two separate results together and see. If we know that this limit at the top here exists, then we deduce that all of the one-sided limits exist and they are all equal to the limit at the top; so they are all identical. In particular, since these two one sided limits exist and are equal then we know that $\lim_{t \to c} \frac{f(t)}{g(t)}$ exists. and is equal to the common value.

Roy left of board

Let me rearrange this a little to give the result we promised you at the start of the programme.

So that's L'Hopital's Rule. Usually the result is stated in terms of x rather than t.

If the functions are differentiable in an openinterval then the rule holds at any point c, in that interval, at which both functions vanish

DAVID BRANNAN:

David and Roy

Well we can now deal with the first of our two problems. The numerator and denominator both vanish at x=11/2. So L'Hopital says that the limit is this. The limit is x tends to 11/2 of this "the derivatives of sin x-e to the cos x" so long as this limit exists. And because the numerator and denominator are both

David and Board 7

continuous at 11/2 and that's 3/1 or 3. Well that was very quick but as I said the second problem is more tricky and I'm going to solve that at the end of the programme. But first let's have a quick resume of how we derived L'Hopital's Rule, followed by a different way of looking at what the Rule says.

D. Brannan (continued)

Frame both panels

David and Board 6

We've had four theorems.

Let's remind ourselves of each of them.

First of all Roy took Rolle's Theorem which said that if f(a) and f(b) =0 then there must be a point inside the interval a,b with a horizontal tangent.

Then Rit demonstrated the Mean Value Theorem which generalises Rolle: where f(a) doesn't necessarily = f(b). There is still a point - k in the interval - at which a tangent must be parallel to the chord.

But those theorems were for functions of the type y=f(x). And here's the next two.

This third step I took brought us immediately closer to quotients by looking at the Mean Value Theorem for curves given in parametric form.

Then Roy from looking at the special case when the parametric curve passed through the origin was able to prove L'Hopital Rule. That states that the limit of our ratio f/g is equal to the limit of the ratio of the derivatives 'if that exists'. Those are the four theorems in the programme and following the same idea of looking at things geometrically let's briefly have another look at Problem 1.

Here are f and g again this time with the tangents drawn in at c=11/2. If you look close to 11/2 you find that the tangents closely approximate the curves and because these triangles have the same base, you find that the ratio of f/g is more or less the same as the ration $\frac{f'}{g'}$ and that looks as though it's 3/1.

Well of course it was the geometrical picture which guided mathematicians like L'Hopital.

MUSIC

It would be intriguing to be able to look over L'Hopital's shoulder to see the way

D. Brannan (continued)

rules such as this were 'proved' in the 17th Century.

MUSIC

ROY NELSON:

(In fancy dress)

Imagine that I have stepped into L'Hopital's shoes for a moment. . . probably it all began with something sketched out on the back of an envelope.

CU O/S writing

He would have started with something like this: something like this: something he already knew. Now of course his notation would have been different from ours, but in essence it would say the same thing that the limit of this ratio = f'(c) and the limit of this ratio = g'(c). What he did next was to change these limits into

CU O/S writing

approximations.

So, he now has approximations to the slopes of the tangents at the point c.

2S Roy and Envelope

It looks very tempting to divide these, one by the other, so get rid of the (x-c)'s to get this approximate value for the ratio and then because f(c) and g(c)

CU O/S wirting

are both zero, he surmised that going back to the limit and get the limit of f/g=f'/g' at the point c.

That was more or less how L'Hopital arrived at his rule. Back in the 20th

2S Roy and Envelope

Century it's known that arguments like this can break down. L'Hopital was lucky.

CU Plaque

His conclusion is correct even if his method of proof leaves something to be desired.

DAVID BRANNAN:

David and Board 2

Well, L'Hopital's rule helped us to evaluate the limit of Problem 1; can it help us here too? If this is our f and this our g we find both vanish at x=0. But, we know what to do, L'Hopital to the rescue.

Surely all we need to do is find the limit $\frac{f'}{g'}$ and evaluate it, that's this

Cloud 2

but evaluating $\frac{f'}{g'}$ gives us an undefined limit again. So are we now stuck?

In fact we aren't. You see we can apply L'Hopital more than once. This is the limit of second derivatives and that is defined at x=0 to be 2/1.

So this limit exists and is equal to 2. You see it's quite acceptable to apply L'Hopital's Rule several times if each time you apply it you find the new quotient also has a numerator and denominator vanishing together. And if you finally reach a ratio that does have a limit you can argue backwards, as here, if the limit of $\frac{f''}{g''}$, exists and = 2 then limit $\frac{f}{g}$ also exists and = 2
And the limit of $\frac{f}{g}$, our original quotient also exists and it equals 2 as well.

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