Diagnostic Quiz for the Open University Mathematics MSc

Contents

Introduction

This quiz is designed to help you see if you have sufficient mathematical background to enrol on the MSc in Mathematics programme, in particular the entry level courses M820 *The Calculus of Variations and Advanced Calculus* and M823 *Analytic Number Theory 1*.

Ideally you should work through all sections of the quiz, identifying any weak areas or gaps in your knowledge. The questions are divided into three sections. The first covers subject areas needed for the Mathematics MSc in general, the second covers the main subject areas needed for M820 and the third the main subject areas needed for M823. The third section also has a few questions that test your aptitude for the more abstract approach required by the pure mathematics topics. The solutions to these questions may not seem obvious at first sight and you may find that you need to think about them for a while. Do not worry about this as doing mathematics at this level often involves spending plenty of time mulling over questions and trying things out before a solution is reached.

Even if you find the questions straightforward you should read through the answers as they may contain extra useful information.

However, you should not worry if you are unfamiliar with one or two subjects (for example, you might not have encountered PDEs before); the important thing is that you can follow the answers and, having done so, would be able to answer similar questions in the future. By the same token, this quiz cannot cover all the subject areas needed for the MSc programme, but if you are able to complete most of the questions here then you ought to be in a good position to succeed on the MSc.

If you find that you do have large gaps in your knowledge then you are advised either to study one of our undergraduate modules, or to do some background reading prior to starting the MSc. Suggestions are given at the end of the quiz, before the answers.

If after working through the quiz you would like to discuss your study plans then you should contact mathstats-support@open.ac.uk .

1 General background

1.1 Differentiation

Exercise 1.1.1

(a) If $x + e^x = t$, find dx/dt and d^2x/dt^2.

(b) If $z = xe^{-y}$, $x = \cosh t$ and $y = \cos t$, find dz/dt.

(c) If $z = (x + y)^5$ and $y = \sin 10x$, find dz/dx.

Exercise 1.1.2

(a) If $w = (r\cos\theta)^{r\sin\theta}$, for $r > 0$ and $0 \leq \theta < \pi/2$, find $\partial w/\partial\theta$.

(b) Let $u = f(x, y)$. Given $x = e^s \cos t$ and $y = e^s \sin t$, find expressions for $\partial u/\partial s$ and $\partial u/\partial t$ in terms of $\partial u/\partial x$ and $\partial u/\partial y$. Hence prove that

$$\frac{\partial u}{\partial y} = e^{-s}\left(\sin t\frac{\partial u}{\partial s} + \cos t\frac{\partial u}{\partial t}\right).$$

Exercise 1.1.3

If $w = f(ax + by)$, where a and b are constants, show that $b\dfrac{\partial w}{\partial x} - a\dfrac{\partial w}{\partial y} = 0$.
(Hint: Let $z = ax + by$.)

Exercise 1.1.4

If $z = \dfrac{1}{x} f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = 0$.

Exercise 1.1.5
Find the Taylor series to second order about $x = -1$, $y = 1$ for the function

$$f(x, y) = x^2 + y^3 + \exp(xy).$$

Exercise 1.1.6
Locate and classify the stationary points of the function

$$f(x, y) = x^3 - 12x - y^3 + 3y.$$

1.2 Integration

Exercise 1.2.1

Evaluate the following integrals:

(a) $\int_0^\infty xe^{-x^2} dx$;

(b) $\int \frac{x}{x^2+3} dx$;

(c) $\int \frac{x^3+3x-2}{x^2-3x+2} dx$, for $x > 2$;

(d) $\int e^{2x} \cos x \, dx$;

(e) $\int \sqrt{(1-4x^2)} \, dx$, for $|x| \leq 1/2$ (hint: use the substitution $2x = \sin\theta$);

(f) $\int_2^3 \frac{x+1}{\sqrt{(x^2-4)}} dx$.

Exercise 1.2.2

Given that

$$I_{p,n} = \int_0^1 (1-x)^p x^n dx,\ p \geq 0,\ n \geq 0,$$

show that $I_{p,n} = I_{n,p}$.

Prove that, for $p \geq$

1,

$$(n+1)I_{p,n} = pI_{p-1,n+1}$$

and also that

$$(p+n+1)I_{p,n} = pI_{p-1,n}.$$

Hence prove that, if p and n are positive integers,

$$I_{p,n} = \frac{p!n!}{(p+n+1)!}.$$

1.3 Linear Algebra

Exercise 1.3.1

(a) Show that if A and B are any two square matrices of the same size, then $(AB)^{-1} = B^{-1}A^{-1}$.

Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$.

(b) Find A^{-1} and B^{-1}.

(c) Verify that $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

(d) Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 1.3.2
Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \text{[illegible]} & & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Exercise 1.3.3
Evaluate the Vandermonde determinant

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \ldots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

1.4 Complex Numbers

Exercise 1.4.1
Find the roots of $z^3 - 7z^2 + 31z - 25 = 0$.

Exercise 1.4.2
Use the polar forms of $1 + i$ and $\sqrt{3} - i$ to evaluate

$$\frac{(1+i)^6}{(\sqrt{3} - i)^3},$$

giving your answer in Cartesian form.

2 Applied Mathematics

2.1 Integration

Exercise 2.1.1
The Fourier Transform $\hat{f}(\omega)$ of a function $f(x)$ may be defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

Using the result $\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2}\, dt = 1$, find the Fourier Transform of $f(x) = e^{-x^2/(2\sigma^2)}$.
(Hint: you might find it helpful to complete the square in the exponent.)

Exercise 2.1.2
By changing the order of integration, show that

$$\int_0^x \left(\int_0^t f(p)\, dp \right) dt = \int_0^x (x - p) f(p)\, dp.$$

Exercise 2.1.3
Consider

$$S[y] = \int_a^b \left(\int_a^b K(s, t) y(s) y(t)\, dt \right) ds,$$

where a and b are constants, y is a function of a single variable and K is a function of two variables. ($S[y]$ is termed a *functional* , with the square brackets around the y emphasising the fact that S depends on the choice of the function y used to evaluate the integral.) Assuming that the order of integration may be interchanged, show that

$$\frac{d}{d\varrho} S[y + \varrho h] \Big|_{\varrho=0} = \int_a^b h(s) \left(\int_a^b [K(s, t) + K(t, s)]\, y(t)\, dt \right) ds.$$

2.2 Differential Equations

2.2.1 First Order ODEs

Exercise 2.2.1
Find the general solutions of the following differential equations:

(a) $xy \frac{dy}{dx} = y^2 + 1$;

(b) $y \frac{dy}{dx} = e^{x+y^2}$;

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} - \tan\dfrac{y}{x}$. (Hint: use the substitution $y = xu(x)$.)

Exercise 2.2.2

Find the particular solution of the following initial-value problem:

$t\dfrac{dy}{dt} + 2y = t^2,\ y(1) = 1.$

2.2.2 Higher Order ODEs

Exercise 2.2.3

Find the general solutions of the following differential equations:

(a) $\dfrac{d^2y}{dx^2} - 4y = x^2;$

(b) $4x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - y = 0$; (Hint: use the substitution $x = e^t$;)

(c) $\dfrac{d^4y}{dx^4} - 4y = 8.$

Exercise 2.2.4

Find the solutions of the following problems:

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0, \quad y(0) = 0,\ y'(0) = 1,$

(b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 5,\ y(0) = 1,\ y(\pi) = 1.$

2.2.3 PDEs

Exercise 2.2.5

Find the ordinary differential equation satisfied by $f(r)$ if $\dfrac{1}{r}f(r)\cos\omega t$ is a solution of the PDE

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where ω and c are constants.

Hence show that a solution of this PDE is

$$u = \frac{1}{r}(A\cos nr + B\sin nr)\cos\omega t,$$

where $n = \omega/c$, and A and B are arbitrary constants.

For a particular solution the following three conditions all hold: u is finite at $r = 0$ for all t, $\partial u/\partial r = 0$ at $r = a$ for all t, and u is not identically zero. Show that $A = 0$ and obtain the equation which must be satisfied by ω.

Exercise 2.2.6

The error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\Pi}} \int_0^x e^{-u^2} du.$$

Show that

$$u(x, t) = A\left(1 - \mathrm{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right)$$

is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$$

where A and k are constants.

3 Pure Mathematics

3.1 Mathematical Proof

Exercise 3.1.1
Using mathematical induction prove that, if f is the function

$$f(x) = xe^x,$$

then, for all $n = 1, 2, \ldots$, the nth derivative of f is given by the formula

$$f^{(n)}(x) = (n + x)e^x.$$

Exercise 3.1.2
Prove by induction that $3^n < n!$ for all integers $n \geq 7$.

Exercise 3.1.3
Use proof by contradiction to prove that there are no integers m and n such that $5m + 15n = 357$.

Exercise 3.1.4
For each of the conditions given in (i)-(v), decide whether it is necessary, sufficient, necessary and sufficient, or neither necessary nor sufficient, in order that $x \equiv y(\text{mod}10)$:

(i) $x - y = 50$, (ii) $x - y = 5$, (iii) $x - y$ is divisible by 10,

(iv) $x - y$ is divisible by 5, (v) $x - y$ is divisible both by 5 and by 2,

(vi) x and y are both even.

Exercise 3.1.5
What is wrong with the following 'proofs':

(a) Prove that $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$.

'Proof'.

Taking sixth powers of each side we have:

$$\begin{aligned} 3^{\frac{1}{3}} > 2^{\frac{1}{2}} \quad &\Rightarrow \quad \left(3^{\frac{1}{3}}\right)^6 > \left(2^{\frac{1}{2}}\right)^6, \\ &\Rightarrow \quad 3^2 > 2^3, \\ &\Rightarrow \quad 9 > 8, \end{aligned}$$

Clearly $9 > 8$ and so we have the result.

(b) Prove that $-i = i$

'Proof'.

$$
\begin{aligned}
\frac{-x}{x} = \frac{x}{-x} \quad &\Leftrightarrow \quad \sqrt{\frac{-x}{x}} = \sqrt{\frac{x}{-x}},\\
&\Leftrightarrow \quad \frac{\sqrt{-1\times x}}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{-1\times x}},\\
&\Leftrightarrow \quad \frac{i\times\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{i\times\sqrt{x}},\\
&\Leftrightarrow \quad i = \frac{1}{i},\\
&\Leftrightarrow \quad i = -i.
\end{aligned}
$$

3.2 Analysis

Exercise 3.2.1

Evaluate the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3}$,

(b) $\lim\limits_{x\to \pi/2}\dfrac{\ln(2-\sin x)}{\ln(1+\cos x)}$,

(c) $\lim\limits_{x\to \infty}\dfrac{\ln x}{\sqrt{x}}$,

(d) $\lim\limits_{x\to 0} x\ln 2x$.

Exercise 3.2.2

Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = 0$.
Show, from the definition of the limit of a sequence, that $\lim\limits_{n\to\infty} a_n b_n = 0$.

Exercise 3.2.3

For $|z| = 2$, prove that

(a) $|z^2 - 4z - 3| \leq 15$,

(b) $|z^2 - 7| \geq 3$,

(c) $|z^2 + 2| \geq 2$.

Hence find a number M such that

$$\left|\frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)}\right| \leq M, \text{ for } |z| = 2.$$

Exercise 3.2.4

The Squeeze Rule for continuity says that a function f is continuous at a if f, g and h are functions defined on an open interval I, $a \in I$ and the following three criteria hold:

(i) $g(x) \leq f(x) \leq h(x), x \in I$, (ii) $g(a) = f(a) = h(a)$ and (iii) g and h are continuous at a.

Use the Squeeze Rule to prove that the following function is continuous at 0:

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Exercise 3.2.5

Give examples of functions f and g of a real variable with $\lim_{x\to\infty} \frac{f}{g} = 1$ but $\lim_{x\to\infty} \frac{e^f}{e^g} \neq 1$.

3.3 Algebra

Exercise 3.3.1

Let G be a group. Let $H = \langle ghg^{-1}h^{-1} : g, h \in G \rangle$. In other words H is the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$ where g and h are in G.
H is known as the *commutator subgroup* (or the *derived group*) of G.

(a) Show that H is a normal subgroup of G.

(b) Give an example of a non trivial group M with a trivial commutator subgroup.

(c) Give an example of a group N with a non-trivial commutator subgroup.

(d) What can you say about the quotient group $\frac{G}{H}$ (also known as the factor group)?

Exercise 3.3.2

Show that the set of n^{th} roots of unity forms a group under multiplication.

3.4 Working with Definitions

This section is designed to test how well you can work with abstract definitions. Do not worry if you have not come across these concepts before as all of the information that you need in order to do the questions is contained within the questions.
The first two questions in this section both make use of the definition below.

Definition

Let X be any set and let T be a collection of subsets of X. T is called a *topology on* X if the following conditions are satisfied:

(T1) T contains both X and $\emptyset$.
(T2) If A and B are in T then so is $A \cap B$.
(T3) If $\{A_i\}$ is any collection of subsets in T then their union is also in T.

Exercise 3.4.1

(a) Let X be any set. Give an example of a collection, T, of subsets which will form a topology on X.
Hint: think about the minimum number of sets that you will need in any topology.

(b) What difference will it make if we replace (T3) with (T3′) below:

(T3′) If A and B are in T then so is $A \cup B$.

Exercise 3.4.2
Let X be the set $\{a, b, c\}$

(a) Show that $T = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ is a topology on X.

(b) Show that $T' = \{X, \emptyset, \{a, c\}, \{a, b\}\}$ is not a topology on X.

Exercise 3.4.3
A *monoid* is a set S together with a binary operation, $\circ$, which satisfies the following axioms:

(M1) $\circ$ is a closed operation. In other words we have $s \circ t \in S$ for all s and $t \in S$.
(M2) $\circ$ is associative. In other words we have $r \circ (s \circ t) = (r \circ s) \circ t$ for all r, s and $t \in S$.
(M3) We have an identity. In other words we have an $e \in S$ where $e \circ s = s \circ e = s$ for all $s \in S$.

(a) Let $C[0, 1]$ denote the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$. Explain why it does not make sense to try to create a monoid using $C[0, 1]$ as the set and composition of functions as the binary operation.

(b) Let $C(\mathbb{R})$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that we can put a monoid structure onto $C(\mathbb{R})$ using composition of functions as our operation, stating any results that you assume.

Exercise 3.4.4
Use the numbers $1, -1, i$ and $-i$ to fill in a 4×4 grid so as to satisfy the following conditions:

(1) the sum of the numbers in the top row is equal to 4,
(2) the sum of the numbers in all other rows is zero,
(3) the sum of the numbers in the left hand column is equal to 4,
(4) the sum of the numbers in all other columns is zero,
(5) each row is different to every other row and each column is different to every other column.

4 What to do now?

If you have been able to make reasonable attempts at most of the questions in this quiz then you should be in a strong position to make good progress on the MSc in Mathematics.

If however you feel that there are gaps in your knowledge then you would be well advised to study material from one or more of the OU's undergraduate modules first. Alongside each solution in the quiz we have listed the OU undergraduate module which covers the relevant mathematics in that question. The modules that are particularly suitable preparation are listed below. Please note that there is an assumption that you are fluent and confident with the mathematics in some or all of the following:

- M208 *Pure mathematics,*
- MST210 *Mathematical methods, models and modelling* (or its predecessor MST209), or MST224 *Mathematical methods*,
- MST326 *Mathematical methods and fluid mechanics,*
- MS324 *Waves, diffusion and variational principles,*
- MS325 *Computer algebra, chaos and simulations,*
- M337 *Complex analysis,*
- M303 *Further pure mathematics* or at least two of its predecessors M381, M336 or M338.

Some material from these modules is available from the OpenLearn website at www.open.edu/openlearn.

Alternatively you might prefer to do some background reading and the list on the next page contains some suggestions. These are not recommendations but are some suggestions from MSc tutors. Some are elementary, some more advanced.

As stated in the introduction to this quiz, if you have any further questions as to your suitability for the MSc in Mathematics then please contact mathstats-support@open.ac.uk.

General Background and methods

- Kreyszig *Advanced Engineering Mathematics* (Wiley)
- Boas *Mathematical Methods in the Physical Sciences* (Wiley)
- Matthews *Vector Calculus* (Springer)
- Riley, Hobson and Bence *Mathematical Methods for Physics and Engineering* (Cambridge)

Differential Equations

- Bronson *Differential equations* (Schaum)

Complex Analysis

- Priestly *Introduction to Complex Analysis* (Oxford)
- Stewart *Complex Analysis: The Hitchhiker's guide to the Plane* (CUP)

Metric and topological Spaces

- Sutherland *Introduction to Metric and Topological Spaces* (OUP)

Analysis

- Binmore *Mathematical Analysis, a straightforward approach* (Cambridge)
- Phillips *An Introduction to Analysis and Integration Theory* (Dover)

Group theory

- Jordan and Jordan *Groups* (Butterworth - Heinemann)

Number theory (analytic)

- W. J. LeVeque *Fundamentals of Number Theory* (Dover)

5 Answers

Differentiation

If you have problems attempting the questions in this section then you should consider studying MST210 and/or MST326.

Solution for Exercise 1.1.1

(a) Differentiating implicitly with respect to t we have

$$\frac{dx}{dt} + e^x \frac{dx}{dt} = 1, \qquad (1)$$

from which we obtain

$$\frac{dx}{dt} = \frac{1}{1 + e^x}.$$

Differentiating (1) again with respect to t, we obtain

$$\frac{d^2x}{dt^2} + e^x \frac{d^2x}{dt^2} + e^x \left(\frac{dx}{dt}\right)^2 = 0,$$

from which we have

$$\frac{d^2x}{dt^2} = \frac{-e^x (dx/dt)^2}{1 + e^x} = \frac{-e^x}{(1 + e^x)^3}.$$

(b) The easiest way is probably to use the chain rule. We have

$$\frac{\partial z}{\partial x} = e^{-y}, \frac{\partial z}{\partial y} = -xe^{-y}, \frac{dx}{dt} = \sinh t \text{ and} \frac{dy}{dt} = -\sin t,$$

so the chain rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^{-y}\sinh t + xe^{-y}\sin t \\ &= e^{-\cos t}\sinh t + \cosh t \sin t e^{-\cos t}.\end{aligned}$$

(Note that this question can also be answered using direct substitution; substitute for x and y to obtain z as a function of t, and then differentiate that directly.)

(c) Again although this could be answered using direct substitution we will use the chain rule, this time with

$$\frac{\partial z}{\partial x} = 5(x + y)^4, \frac{\partial z}{\partial y} = 5(x + y)^4 \text{ and } \frac{dy}{dx} = 10 \cos 10x.$$

So

$$\begin{aligned}\frac{dz}{dx} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 5(x + y)^4[1 + 10 \cos 10x] \\ &= 5(x + \sin 10x)^4[1 + 10 \cos 10x].\end{aligned}$$

Solution for Exercise 1.1.2

(a) Taking logs, we obtain $\ln w = r\sin\theta \ln(r\cos\theta)$. Differentiating implicitly gives

$$\frac{1}{w}\frac{\partial w}{\partial\theta} = r\sin\theta\left(\frac{-r\sin\theta}{r\cos\theta}\right) + r\cos\theta\ln(r\cos\theta),$$

so that

$$\frac{\partial w}{\partial\theta} = (r\cos\theta)^{r\sin\theta}\left(r\cos\theta\ln(r\cos\theta) - \frac{r\sin^2\theta}{\cos\theta}\right).$$

(b) Using the chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\frac{\partial u}{\partial y}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = -e^s\sin t\frac{\partial u}{\partial x} + e^s\cos t\frac{\partial u}{\partial y}.$$

Hence

$$\begin{aligned} e^{-s}\frac{\partial u}{\partial s} &= \cos t\frac{\partial u}{\partial x} + \sin t\frac{\partial u}{\partial y}, \\ e^{-s}\frac{\partial u}{\partial t} &= \cos t\frac{\partial u}{\partial y} - \sin t\frac{\partial u}{\partial x}; \end{aligned}$$

multiplying the first of these by $\sin t$ and the second by $\cos t$ and then adding them together gives

$$\frac{\partial u}{\partial y} = e^{-s}\left(\sin t\frac{\partial u}{\partial s} + \cos t\frac{\partial u}{\partial t}\right).$$

Solution for Exercise 1.1.3

Following the hint we let $z = ax + by$ so that $w = f(z)$. Then

$$\frac{\partial w}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = a\frac{dw}{dz},$$

and similarly $\partial w/\partial y = b\, dw/dz$. Hence

$$b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = ab\frac{dw}{dz} - ab\frac{dw}{dz} = 0.$$

Solution for Exercise 1.1.4

Using the product and chain rules, and letting $u = y/x$ so that $z = 1/x\, f(u)$, we have

$$\frac{\partial z}{\partial x} = \frac{1}{x}\frac{df}{du}\frac{\partial u}{\partial x} + f(u)\left(\frac{-1}{x^2}\right) = -\frac{y}{x^3}f'(u) - \frac{1}{x^2}f(u)$$

and

$$\frac{\partial z}{\partial y} = \frac{1}{x}\frac{df}{du}\frac{\partial u}{\partial y} = \frac{1}{x^2}f'(u).$$

Hence

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+z=-\frac{y}{x^2}f'(u)-\frac{1}{x}f(u)+\frac{y}{x^2}f'(u)+\frac{1}{x}f(u)=0.$$

Solution for Exercise 1.1.5

We differentiate $f(x,y)=x^2+y^3+\exp(xy)$ and evaluate the derivatives at $x=-1,\ y=1$. This gives

$$\begin{aligned}
f_x &= 2x+y\exp(xy),\ f_x(-1,1)=-2+\exp(-1);\\
f_y &= 3y^2+x\exp(xy),\ f_y(-1,1)=3-\exp(-1);\\
f_{xx} &= 2+y^2\exp(xy),\ f_{xx}(-1,1)=2+\exp(-1);\\
f_{yy} &= 6y+x^2\exp(xy),\ f_{yy}(2,\ 1)=6+\exp(-1);\\
f_{xy} &= \exp(xy)+xy\exp(xy),\ f_{xy}(-1,\ 1)=0.
\end{aligned}$$

The general formula for the Taylor series for two variables (x, y) about the point (a, b) is

$$f(x,y)=\sum_{n=0}^{\infty}\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^n f(a,b);$$

hence the required Taylor series (to order two) is given by

$$\begin{aligned}
f(x,y) &= f(-1,1)+f_x(-1,1)(x+1)+f_y(-1,1)(y-1)\\
&\quad +f_{xx}(-1,1)(x+1)^2/2!+f_{xy}(-1,1)(x+1)(y-1)\\
&\quad +f_{yy}(-1,1)(y-1)^2/2!\\
&= 2+\exp(-1)+(-2+\exp(-1))(x+1)+(3-\exp(-1))(y-1)\\
&\quad +(2+\exp(-1))(x+1)^2/2\\
&\quad +(6+\exp(-1))(y-1)^2/2.
\end{aligned}$$

Solution for Exercise 1.1.6

We have $f_x = 3x^2 - 12$ and $f_y = -3y^2 + 3$, so the stationary points (found by putting $f_x = f_y = 0$) are at $(x, y) = (2, 1), (2, -1), (-2, 1)$ and $(-2, -1)$.

Also, $f_{xx} = 6x$, $f_{xy} = 0$ and $f_{yy} = -6y$.

Thus at $(2, 1)$ and at $(-2, -1)$ we have $f_{xx}f_{yy} - f_{xy}^2 = -36xy = -72 < 0$, and these are saddle points.

At $(2, -1)$ and at $(-2, 1)$ we have $f_{xx}f_{yy} - f_{xy}^2 = -36xy = 72 > 0$. Since $f_{xx} > 0$ at $(2, -1)$ this is a local minimum; since $f_{xx} < 0$ at $(-2, 1)$ this is a local maximum.

Integration

The material in this section is covered in the undergraduate modules MST124 and MST125.

Solution for Exercise 1.2.1

In the following answers, c is an arbitrary real constant.

(a) $\int_0^\infty xe^{-x^2}dx = \left[\frac{e^{-x^2}}{-2}\right]_0^\infty = \frac{1}{2}$;

(b) $\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \frac{1}{2}\ln(x^2+3)+c$;

(c) $\int \frac{x^3+3x-2}{x^2-3x+2}dx = \int\left(x+3+\frac{10x-8}{(x-1)(x-2)}\right)dx = \int\left(x+3+\frac{12}{x-2}-\frac{2}{x-1}\right)dx$

$= \frac{x^2}{2} + 3x + 12\ln(x-2) - 2\ln(x-1) + c$;

(d) This question can be solved using integration by parts. However, an alternative neater (and arguably shorter) method is as follows:

$$\int e^{2x}\cos x\,dx = \mathrm{Re}\int e^{2x}e^{ix}dx = \mathrm{Re}\int e^{(2+i)x}dx = \mathrm{Re}\,\frac{1}{2+i}e^{(2+i)x}+c$$
$$= \mathrm{Re}\,\frac{2-i}{5}e^{2x}e^{ix}+c = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + c;$$

(e) Let $2x = \sin\theta$; then

$\int\sqrt{(1-4x^2)}dx = \int\sqrt{(1-\sin^2\theta)}\frac{1}{2}\cos\theta\,d\theta = \frac{1}{2}\int\cos^2\theta\,d\theta = \frac{1}{4}\int(1+\cos 2\theta)\,d\theta$

$= \frac{1}{4}(\theta+\frac{1}{2}\sin 2\theta)+c = \frac{1}{4}(\theta+\sin\theta\cos\theta)+c = \frac{1}{4}(\arcsin(2x)+2x\sqrt{(1-4x^2)})+c$;

(f) Observe there is a singularity at $x=2$ so the integral is improper. However we will integrate as normal and see what happens.

Let $x = 2\cosh u$; then

$\int_2^3 \frac{x+1}{\sqrt{(x^2-4)}}dx = \int_{x=2}^{x=3}\frac{2\cosh u+1}{\sqrt{(4\cosh^2 u-4)}}2\sinh u\,du = \int_{x=2}^{x=3}(2\cosh u+1)\,du$

$= [2\sinh u+u]_{x=2}^{x=3} = \left[2\sqrt{\left((x/2)^2-1\right)}+\operatorname{arcosh}\frac{x}{2}\right]_2^3 = \sqrt{5}+\operatorname{arcosh}(3/2)$.

The evaluation worked because the primitive was defined and continuous at $x=2$.

Solution for Exercise 1.2.2

Using the substitution $u = 1-x$,

$$\begin{aligned} I_{n,p} &= \int_0^1 (1-x)^n x^p dx = \int_1^0 u^n(1-u)^p(-du) = \int_0^1 u^n(1-u)^p du \\ &= \int_0^1 x^n(1-x)^p dx = I_{p,n} \end{aligned}$$

as required.

For the next part of the question,

$$I_{p,n} = \int_0^1 (1-x)^p x^n \, dx = \left[(1-x)^p \frac{x^{n+1}}{n+1}\right]_0^1 + \int_0^1 \frac{x^{n+1}}{n+1} p(1-x)^{p-1} dx,$$

so that

$$(n+1)I_{p,n} = pI_{p-1,n+1} \tag{2}$$

as required. Using this result, we then have

$$\begin{aligned}(p+n+1)I_{p,n} &= pI_{p,n} + (n+1)I_{p,n} = pI_{p,n} + pI_{p-1,n+1} \\ &= p\int_0^1 \left((1-x)^p x^n + (1-x)^{p-1} x^{n+1}\right) dx \\ &= p\int_0^1 (1-x)^{p-1} x^n [1 - x + x]\, dx\end{aligned}$$

so that

$$(p+n+1)I_{p,n} = pI_{p-1,n} \tag{3}$$

as required.

From (3) we have

$$I_{p,n} = \frac{p}{(p+n+1)} I_{p-1,n}.$$

From (2),

$$I_{p-1,n+1} = \frac{(n+1)}{p} I_{p,n} = \frac{(n+1)}{(p+n+1)} I_{p-1,n},$$

which is equivalent to

$$I_{p-1,n} = \frac{n}{p+n} I_{p-1,n-1}.$$

So overall this gives

$$\begin{aligned}I_{p,n} &= \frac{p}{(p+n+1)}\frac{n}{(p+n)} I_{p-1,n-1} \\ &= \frac{p \quad n \quad p-1 \quad n-1}{(p+n+1)\,(p+n)\,(p+n-1)\,(p+n-2)} I_{p-2,n-2} \\ &= \frac{p!n!}{(p+n+1)!} I_{0,0},\end{aligned}$$

and $I_{0,0} = \int_0^1 dx = 1$ gives the final result.

Linear algebra

If you have problems attempting the questions in this section then you should consider studying M208.

Solution for Exercise 1.3.1

(a) We have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

(we need to check both of these as matrix multiplication is not commutative).

Hence $(AB)^{-1} = B^{-1}A^{-1}$.

(b) Since $\det A = 2$, we have

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix}.$$

Similarly

$$B^{-1} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

(c)

$$(A+B)^{-1} = \begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -\frac{5}{3} \end{pmatrix}$$

and

$$A^{-1} + B^{-1} = \begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & 1 \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}.$$

Thus $(A+B)^{-1} \quad A^{-1} + B^{-1}$.

(d)

$$(AB)^{-1} = \left(\begin{pmatrix} 2 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 4 & 4 \\ 7 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{pmatrix}$$

and

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{pmatrix}.$$

Thus $(AB)^{-1} = B^{-1}A^{-1}$.

Solution for Exercise 1.3.2

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0.$$

Expanding the determinant by the first row gives

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+1) = 0,$$

so $1-\lambda = 0$ or $\lambda^2 + 1 = 0$. Thus the eigenvalues are $\lambda = 1$, $\lambda = i$ and $\lambda = -i$.

The eigenvector equations are

$$\begin{aligned}
(1-\lambda)x_1 \qquad\qquad &= 0,\\
-\lambda x_2 + \ \ x_3 &= 0,\\
-x_2 - \lambda x_3 &= 0.
\end{aligned}$$

When $\lambda = 1$ the eigenvector equations become $0 = 0,\ -x_2 + x_3 = 0$ and $-x_2 - x_3 = 0$, which reduce to the equations $x_2 = 0$ and $x_3 = 0$. There is no constraint on x_1, so we may choose it as we please. Hence a corresponding eigenvector is $[1\ 0\ 0]^T$.

When $\lambda = i$ the eigenvector equations become $(1 - i)x_1 = 0,\ -ix_2 + x_3 = 0$ and $-x_2 - ix_3 = 0$, which reduce to the equations $x_1 = 0$ and $ix_2 = x_3$. It follows that a corresponding eigenvector is $[0\ 1\ i]^T$.

When $\lambda = -i$ the eigenvector equations become $(1 + i)x_1 = 0,\ ix_2 + x_3 = 0$ and $-x_2 + ix_3 = 0$, which reduce to the equations $x_1 = 0$ and $-ix_2 = x_3$. It follows that a corresponding eigenvector is $[0\ 1\ -i]^T$.

Solution for Exercise 1.3.3

We subtract x_1 times the penultimate row from the last row, then x_1 times the $(n-2)$-th row from the penultimate row, then x_1 times the $(n-3)$-th row from the $(n-2)$-th row, etc., ending with x_1 times the first row from the second row. This gives us

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1\\
0 & x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1\\
0 & x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \ldots & x_n^2 - x_1x_n\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
0 & x_2^{n-2} - x_1x_2^{n-3} & x_3^{n-2} - x_1x_3^{n-3} & \ldots & x_n^{n-2} - x_1x_n^{n-3}\\
0 & x_2^{n-1} - x_1x_2^{n-2} & x_3^{n-1} - x_1x_3^{n-2} & \ldots & x_n^{n-1} - x_1x_n^{n-2}
\end{vmatrix}.$$

Evaluating this using the first column and factorising each element gives

$$V_n(x_1, x_2, \ldots, x_n) = \begin{vmatrix}
x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1\\
x_2(x_2 - x_1) & x_3(x_3 - x_1) & \ldots & x_n(x_n - x_1)\\
\ldots & \ldots & \ldots & \ldots\\
x_2^{n-3}(x_2 - x_1) & x_3^{n-3}(x_3 - x_1) & \ldots & x_n^{n-3}(x_n - x_1)\\
x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \ldots & x_n^{n-2}(x_n - x_1)
\end{vmatrix}.$$

Factoring out from the first column the common factor $x_2 - x_1$, from the second column $x_3 - x_1$, etc., ending with $x_n - x_1$ from the $(n-1)$-th column, we obtain

$$\begin{vmatrix}
1 & 1 & \ldots & 1\\
x_2 & x_3 & \ldots & x_n\\
x_2^2 & x_3^2 & \ldots & x_n^2\\
\ldots & \ldots & \ldots & \ldots
\end{vmatrix}.$$

$$V_n(x_1, x_2, \ldots, x_n) = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1)$$

$$x_2^{n-2} \quad x_3^{n-2} \quad \ldots \quad x_n^{n-2}$$

Following the same procedures eventually results in

$$V_n(x_1, x_2, \ldots, x_n) = \Pi_{n\geq k>i\geq 1}(x_k - x_i).$$

Complex numbers

If you have problems attempting the questions in this section then you should consider studying M208 and/or M337.

Solution for Exercise 1.4.1

By inspection we see that $z = 1$ is one root of the given cubic, and so we factorise the cubic as follows:

$$z^3 - 7z^2 + 31z - 25 = (z - 1)(z^2 - 6z + 25).$$

Solving the quadratic equation $z^2 - 6z + 25 = 0$ gives the other two roots as $z = 3 \pm 4i$.

Solution for Exercise 1.4.2

We have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

so by de Moivre's Theorem

$$(1 + i)^6 = 2^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right).$$

Similarly, as

$$\sqrt{3} - 1 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right),$$

we have

$$(\sqrt{3} - i)^{-3} = 2^{-3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

Hence

$$\begin{aligned}\frac{(1 + i)^6}{(\sqrt{3} - i)^3} &= 2^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \times 2^{-3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\\ &= \cos 2\pi + i\sin 2\pi = 1.\end{aligned}$$

Applied Mathematics

Integration

Solution for Exercise 2.1.1

If you have problems attempting this question then you should consider studying MS324.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2} - i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x+\sigma^2 i\omega)^2 + \sigma^4\omega^2\right]}dx,$$

on completing the square.

Now let $y = x + \sigma^2 i\omega$ so that

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} e^{-\frac{\sigma^2\omega^2}{2}}\, dy = \frac{2}{\sqrt{2\pi}} e^{-\frac{\sigma^2\omega^2}{2}} \int_0^{\infty} e^{-\left(\frac{y}{\sqrt{2}\sigma}\right)^2}\, dy \\ &= \sqrt{\frac{2}{\pi}}\, e^{-\frac{\sigma^2\omega^2}{2}} \int_0^{\infty} e^{-t^2}\sqrt{2}\sigma\, dt,\end{aligned}$$

on letting $y = \sqrt{2}\sigma t$. Use of the integral result given in the question gives the Fourier Transform as

$$\hat{f}(\omega) = \sigma e^{-\frac{\sigma^2\omega^2}{2}}.$$

Solution for Exercise 2.1.2

If you have problems attempting this question then you should consider studying MS324.

With $\int_0^x \left(\int_0^t f(p)\, dp\right) dt$, we first integrate with respect to p for p from 0 to t, and then with respect to t for t from 0 to x. If we change the order of integration then we will integrate with respect to t for t from p to x, and then with respect to p for p from 0 to x.

Hence we have

$$\begin{aligned}\int_0^x \left(\int_0^t f(p)\, dp\right) dt &= \int_0^x \left(\int_p^x f(p)\, dt\right) dp = \int_0^x [tf(p)]_{t=p}^{t=x}\, dp \\ &= \int_0^x (x-p)f(p)\, dp.\end{aligned}$$

Solution for Exercise 2.1.3

First note that

$$S[y + \varrho h] = \int_a^b \left(\int_a^b K(s,t)(y(s) + \varrho h(s))(y(t) + \varrho h(t))\, dt\right) ds,$$

so that

$$\frac{d}{d\varrho} S[y + \varrho h] = \int_a^b \left(\int_a^b K(s,t)\,[y(s)h(t) + h(s)y(t) + O(\varrho)]\, dt\right) ds.$$

Taking the limit as $\varrho \to 0$ gives

$$\begin{aligned}\frac{d}{d\varrho}S[y+\varrho h]\bigg|_{\varrho=0} &= \int_a^b\int_a^b K(s,t)\,[y(s)h(t)+h(s)y(t)]\,dt\,ds \\ &= \int_a^b\left(\int_a^b K(s,t)y(s)h(t)\,dt\right)ds + \int_a^b\left(\int_a^b K(s,t)h(s)y(t)\,dt\right)ds.\end{aligned}$$

Now, we can interchange the order of integration for the first term, so that

$$\int_a^b\left(\int_a^b K(s,t)y(s)h(t)\,dt\right)ds = \int_a^b\left(\int_a^b K(s,t)y(s)h(t)\,ds\right)dt,$$

and the second term can be written as follows:

$$\int_a^b\left(\int_a^b K(s,t)h(s)y(t)\,dt\right)ds = \int_a^b\left(\int_a^b K(t',s')h(t')y(s')\,ds'\right)dt',$$

where the dummy variable s has been replaced by t' and similarly t has been replaced by s'. Hence

$$\frac{d}{d\varrho}S[y+\varrho h]\bigg|_{\varrho=0} = \int_a^b\left(\int_a^b K(s,t)y(s)h(t)\,ds\right)dt + \int_a^b\left(\int_a^b K(t',s')h(t')y(s')\,ds'\right)dt'.$$

Finally, we can change dummy variable again in the second term, replacing s' by s and t' by t to obtain

$$\begin{aligned}\frac{d}{d\varrho}S[y+\varrho h]\bigg|_{\varrho=0} &= \int_a^b\left(\int_a^b K(s,t)y(s)h(t)\,ds\right)dt + \int_a^b\left(\int_a^b K(t,s)h(t)y(s)\,ds\right)dt \\ &= \int_a^b\left(\int_a^b [K(s,t)y(s)h(t)+K(t,s)h(t)y(s)]\,ds\right)dt,\end{aligned}$$

and as $h(t)$ is independent of s we can bring it out of the s integral to get

$$\frac{d}{d\varrho}S[y+\varrho h]\bigg|_{\varrho=0} = \int_a^b h(t)\left(\int_a^b [K(s,t)+K(t,s)]\,y(s)\,ds\right)dt,$$

which, on changing variables one more time, gives the required result.

Differential Equations

If you have problems attempting the questions in this section then you should consider studying MST210 or MST224 and/or MST326.

Solution for Exercise 2.2.1

(a) Separating the variables and integrating gives $\displaystyle\int \frac{y}{y^2+1}\,dy = \int \frac{1}{x}\,dx$ so that

$$\frac{1}{2}\ln(y^2+1) = \ln x + \ln c,$$

where c is a constant and we use $\ln c$ for the constant of integration to make subsequent manipulations easier. Hence $\ln(y^2 + 1) = \ln(xc)^2$ so that

$$y = \pm\sqrt{(dx^2 - 1)},$$

where d is a constant (equal to c^2).

(b) We can write the differential equation as $y\dfrac{dy}{dx} = e^x e^{y^2}$ so that on separating the variables we get $ye^{-y^2}\,dy = e^x dx$. Integrating, we obtain

$$-\frac{1}{2}e^{-y^2} = e^x + c,$$

so that $y = \pm\sqrt{(-\ln(-2e^x + d))}$ (for c and d both constants).

(c) Following the hint, let $y = xu$ (where u is a function of x). Then the product rule for differentiation gives $y' = xu' + u$, the prime denoting differentiation with respect to x. Substituting this, along with $y = xu$, into the original equation gives $xu' + u = u - \tan u$ so that the original differential equation has transformed into

$$x\frac{du}{dx} = -\tan u.$$

This can be solved by separating the variables:

$$\frac{\cos u}{\sin u}\,du = -\frac{1}{x}\,dx$$

so that $\ln(\sin u) = -\ln x + \ln c = \ln(c/x)$, where c is an arbitrary constant. Hence $u = \arcsin(c/x)$, and so the solution to the original differential equation is

$$y = x\arcsin(c/x).$$

Solution for Exercise 2.2.2

After division by t, the given equation can be written as

$$\frac{dy}{dt} + \frac{2}{t}y = t. \qquad (4)$$

(To avoid division by zero, we can set $t > 0$, say, which is consistent with the given initial condition.)

The integrating factor is

$$p = \exp\int\frac{2}{t}\,dt = \exp(2\ln t) = t^2,$$

and multiplying (4) through by p gives

$$t^2\frac{dy}{dt} + 2ty = t^3.$$

Thus the differential equation can be written as

$$\frac{d}{dt}\left(t^2 y\right) = t^3,$$

and on integrating we obtain the general solution $t^2y = t^4/4 + c$, or equivalently $y = t^2/4 + ct^{-2}$, where c is an arbitrary constant.

From the initial condition $y(1) = 1$ we have $1 = 1/4 + c$, so $c = 3/4$ and hence the solution of the initial-value problem is

$$y = \frac{1}{4}(t^2 + 3t^{-2}).$$

Solution for Exercise 2.2.3

(a) The auxiliary equation is $m^2 - 4 = 0$ which has solutions $m = \pm 2$ and so the complementary function is $y = Ae^{2x} + Be^{-2x}$ or $y = C\sinh(2x) + D\cosh(2x)$ where A, B, C and D are arbitrary constants (either form is perfectly acceptable).

For the particular solution try $y = ax^2 + bx + c$ so that $y'' = 2a$ and on substituting these into the original differential equation we obtain

$$2a - 4(ax^2 + bx + c) = x^2.$$

On equating coefficients we get $4a = -1$ so that $a = -1/4$, $2a - 4c = 0$ so that $c = -1/8$, and $b = 0$. Hence the general solution is

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{4}x^2 - \frac{1}{8}$$

or

$$y = C\sinh(2x) + D\cosh(2x) - \frac{1}{4}x^2 - \frac{1}{8}.$$

(b) For $x = e^t$, note that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

and that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}.$$

Substituting these into the original differential equation and simplifying results in

$$4\frac{d^2y}{dt^2} - y = 0,$$

the general solution of which is $y = Ae^{t/2} + Be^{-t/2}$, where A and B are arbitrary constants.

Hence the general solution of the original differential equation is

$$y = A\sqrt{x} + \frac{B}{\sqrt{x}}.$$

(c) Here the auxiliary equation is $m^4 - 4 = 0$, which has solutions $m = \pm\sqrt{2}, \pm\sqrt{2}i$. So the complementary function is

$$y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x),$$

where A, B, C and D are arbitrary constants.
Alternatively this solution may be written as

$$y = A\sinh(\sqrt{2}x) + B\cosh(\sqrt{2}x) + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x).$$

For the particular integral, let $y = a$. Then $y^{(iv)} = 0$ and so $-4a = 8$. Hence the general solution is

$$y = A\sinh(\sqrt{2}x) + B\cosh(\sqrt{2}x) + C\sin(\sqrt{2}x) + D\cos(\sqrt{2}x) - 2.$$

Solution for Exercise 2.2.4

(a) The auxiliary equation is $m^2 - 3m + 2 = 0$, which has roots $m = 1$ and $m = 2$. Hence the complementary function is $y = Ce^x + De^{2x}$, where C and D are arbitrary constants. Since the differential equation is homogeneous, this is also the general solution.

Differentiating the general solution we obtain $y' = Ce^x + 2De^{2x}$, so the initial conditions $y(0) = 0$ and $y'(0) = 1$ give

$$0 = Ce^0 + De^0 = C + D,$$
$$1 = Ce^0 + 2De^0 = C + 2D.$$

Solving these gives $C = -1, D = 1$, so the required solution to this initial-value problem is

$$y = -e^x + e^{2x}.$$

(b) The auxiliary equation is $m^2 + 4m + 5 = 0$, which has solutions $m = -2 \pm i$. So the complementary function is

$$y_c = e^{-2x}(C\cos x + D\sin x),$$

for C and D arbitrary constants.
To find a particular integral, try $y = a$. Substituting this into the differential equation gives $a = 1$, so a particular integral is $y_p = 1$ and the general solution is

$$y = e^{-2x}(C\cos x + D\sin x) + 1.$$

The boundary conditions are $y(0) = 1$ and $y(\pi) = 1$; substituting each of these in turn into the general solution gives

$$1 = e^0(C\cos 0 + D\sin 0) + 1 = C + 1,$$
$$1 = e^{-2\pi}(C\cos\pi + D\sin\pi) + 1 = -Ce^{-2\pi} + 1.$$

Both of these reduce to $C = 0$, but D can take any value, so any solution of the form

$$y = De^{-2x}\sin x + 1$$

satisfies the differential equation and the boundary conditions. This is therefore an example of a boundary-value problem which does not have a unique solution; there is an infinite family of possible solutions.

Solution for Exercise 2.2.5

With $u = \frac{1}{r} f(r) \cos \omega t$, we have

$$\frac{\partial u}{\partial r} = \cos \omega t \left(\frac{1}{r} f'(r) - \frac{1}{r^2} f(r) \right),$$

$$\frac{\partial^2 u}{\partial r^2} = \cos \omega t \left(\frac{1}{r} f''(r) - \frac{2}{r^2} f'(r) + \frac{2}{r^3} f(r) \right),$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\omega^2}{r} f(r) \cos \omega t.$$

Substituting these into the given PDE, dividing through by $\cos \omega t$ and rearranging gives

$$f''(r) + \frac{\omega^2}{c^2} f(r) = 0,$$

which is the required ODE. This has solution $f(r) = A \cos nr + B \sin nr$ (where $n = \omega/c$) so that the required solution to the PDE is

$$u = \frac{1}{r}(A \cos nr + B \sin nr) \cos \omega t.$$

For u to be finite at $r = 0$ requires $A = 0$, so $u = B/r \sin nr \cos \omega t$. Also,

$$\frac{\partial u}{\partial r} = \left(\frac{B}{r} n \cos nr - \frac{B}{r^2} \sin nr \right) \cos \omega t,$$

and for this to be zero at $r = a$ for all t requires

$$\frac{B}{a} n \cos na - \frac{B}{a^2} \sin na = 0;$$

excluding the trivial solution $u \equiv 0$ results in $\tan na = na$, so the required equation for ω is

$$\tan \frac{\omega a}{c} = \frac{\omega a}{c}.$$

Solution for Exercise 2.2.6

If you have problems attempting this question then you should consider studying MS324. From

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

we have

$$\frac{d}{dx}(\operatorname{erf}(x)) = \frac{2}{\sqrt{\pi}} e^{-x^2};$$

hence for $u = A - A \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} \right)$ we have

$$\frac{\partial u}{\partial x} = -\frac{2A}{\sqrt{\pi}} e^{-x^2/(4kt)} \frac{1}{2\sqrt{kt}},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{A}{\sqrt{\pi kt}} \frac{2x}{4kt} e^{-x^2/(4kt)},$$

$$\frac{\partial u}{\partial t} = -\frac{2A}{\sqrt{\pi}} e^{-x^2/(4kt)} \frac{x}{2\sqrt{k}} \frac{-1}{2t\sqrt{t}}.$$

Substituting these into the given PDE gives

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{k}\frac{\partial u}{\partial t} = \frac{2Ax}{4kt\sqrt{\pi kt}} e^{-x^2/(4kt)} - \frac{2Ax}{4kt\sqrt{\pi kt}} e^{-x^2/(4kt)} = 0.$$

Pure Mathematics

Mathematical Proof

If you have problems attempting the questions in this section then you should consider studying M208 and/or M303.

Solution for Exercise 3.1.1

Let $P(n)$ be the proposition $f^{(n)}(x) = (n + x)e^x$.

We first check that $P(1)$ is true. Using the product rule,

$$f^{(1)}(x) = f'(x) = e^x + xe^x = (1 + x)e^x.$$

Hence $P(1)$ is true.

Now we assume that $P(k)$ is true for an integer $k \geq 1$, so that $f^{(k)}(x) = (k + x)e^x$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}(f^{(k)}(x)) = \frac{d}{dx}((k+x)e^x) \\ &= e^x + (k + x)e^x = (k + 1 + x)e^x. \end{aligned}$$

That is we have shown that $P(k) \Rightarrow P(k + 1)$. Hence by induction $P(n)$ is true for $n = 1, 2, \ldots$.

Solution for Exercise 3.1.2

Let $P(n)$ be the statement $3^n < n!$. Then $P(7)$ is true because $3^7 = 2187 < 5040 = 7!$.
(We note that $P(n)$ is false for $n = 1, 2, \ldots, 6$.)

Now assume that $P(k)$ is true for an integer $k \geq 7$, that is that $3^k < k!$.
We wish to deduce that $P(k + 1)$ is true, that is that $3^{k+1} < (k + 1)!$.
Then,

$$\begin{aligned} 3^{k+1} &= 3 \times 3^k < 3 \times k! \text{ (by } P(k)) \\ &< (k + 1)k! \text{ (because } k \geq 7, \text{ and hence } k + 1 \geq 8 > 3) \\ &= (k + 1)!. \end{aligned}$$

That is we have shown that $P(k) \Rightarrow P(k + 1)$. Hence by induction $P(n)$ is true for $n = 7, 8, \ldots$.

Solution for Exercise 3.1.3

Suppose that there exist integers m and n with $5m + 15n = 357$. The left-hand side of this equation is a multiple of 5, so the right-hand side is also. But this is a contradiction, so our supposition must be false. Hence there are no such integers m and n.

Solution for Exercise 3.1.4

(i) This is a sufficient condition. If $x - y = 50$, then x is congruent to y modulo 10. But it is not a necessary condition. For example, $51 \equiv 31 \pmod{10}$, but $51 - 31$ is not equal to 50.

(ii) This condition is neither necessary nor sufficient.

(iii) This condition is both necessary and sufficient that $x \equiv y \pmod{10}$.

(iv) This is a necessary condition, but not sufficient. If $x \equiv y \pmod{10}$, then $x - y$ is divisible by 10, and so must be divisible by 5. However, this condition is not enough to ensure that $x \equiv y \pmod{10}$; for example, we might have $x - y = 15$, and x would then not be congruent to y modulo 10.

(v) This condition is equivalent to that in (iii), and is again a necessary and sufficient condition that $x \equiv y \pmod{10}$.

(vi) This condition is neither necessary nor sufficient.

Solution for Exercise 3.1.5

(a) This statement is true as $3^{\frac{1}{3}} \approx 1.4422$ and $2^{\frac{1}{2}} \approx 1.4142$. However the 'proof' simply states that $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ and uses this to prove that $9 > 8$ (which is, of course, true but doesn't really require proving!). The logic behind this proof is:
$3^{\frac{1}{3}} > 2^{\frac{1}{2}} \Rightarrow \ldots \Rightarrow 9 > 8$. In other words it says that if $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ then we can show that $9 > 8$. However what we actually want to do here is to show that:
if $9 > 8$ then we have $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$.
To correct this proof we need to replace all the $\Rightarrow$s with $\Leftarrow$s (or $\Leftrightarrow$s).

(b) This statement is clearly false. The problem with the 'proof' is that we are implicitly assuming here that squareroot is a map from $\mathbb{C}$ to $\mathbb{C}$ which respects multiplication. It is not possible to define $\sqrt{\ } : \mathbb{C} \to \mathbb{C}$ in a way that does this (try it!).

Analysis

If you have problems attempting the questions in this section then you should consider studying M208.

Solution for Exercise 3.2.1

There is usually more than one way of answering questions about limits; in the following examples we will use L'Hôpital's rule. This method can be applied in certain circumstances, but needs to be used with care.

(a) $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$
$$= \lim_{x\to 0} \frac{2\sec^4 x + 4\sec^2 x \tan^2 x}{6} = \frac{1}{3},$$

(b) $$\lim_{x\to\pi/2} \frac{\ln(2 - \sin x)}{\ln(1+\cos x)} = \lim_{x\to\pi/2} \frac{-\cos x/(2 - \sin x)}{-\sin x/(1+\cos x)} = 0,$$

(c) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{x^{-1/2}/2} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0,$

(d) $\lim_{x\to 0} x \ln 2x = \lim_{x\to 0} \frac{\ln 2x}{1/x} = \lim_{x\to 0} \frac{2/(2x)}{-1/x^2} = \lim_{x\to 0}(-x) = 0.$

Solution for Exercise 3.2.2

From the definition of a limit of a sequence we know that:

for each $\varrho > 0$, there is an integer N_a such that $|a_n| < \varrho$, for all $n > N_a$; for each $\varrho > 0$, there is an integer N_b such that $|b_n| < \varrho$, for all $n > N_b$.

In particular there is an N_a' such that for $n > N_a'$, $|a_n| < 1$.

Let $N = \max\{N_a', N_b\}$ then for $n > N$, $|a_n b_n| = |a_n||b_n| < \varrho$ and so $\lim_{n\to\infty} a_n b_n = 0$ as required.

Solution for Exercise 3.2.3

To answer this question we will use the Triangle Inequality in both its usual form and its backwards form. These forms are, respectively:

$$|z_1 + z_2| \le |z_1| + |z_2| \text{ and } |z_1 - z_2| \ge \big||z_1| - |z_2|\big|,$$

for any $z_1, z_2 \in \mathbb{C}$.

(a) By the Triangle Inequality,

$$|z^2 - 4z - 3| \le |z^2| + |-4z| + |-3| = |z^2| + 4|z| + 3,$$

so for $|z| = 2$ we have

$$|z^2 - 4z - 3| \le 4 + 8 + 3 = 15.$$

(b) By the backwards form of the Triangle Inequality, we have

$$|z^2 - 7| \ge \big||z|^2 - 7\big|.$$

So for $|z| = 2$ we have

$$|z^2 - 7| \ge |4 - 7| = 3.$$

(c) Similarly,

$$|z^2 + 2| \ge \big||z|^2 - 2\big| = |4 - 2| = 2.$$

Hence we can write

$$\left|\frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)}\right| = \frac{|z^2 - 4z - 3|}{|z^2 - 7|\,|z^2 + 2|} \le \frac{15}{3 \times 2} = \frac{5}{2},$$

and so we can take $M = 5/2$.

Solution for Exercise 3.2.4

Since

$$-1 \le \cos(1/x^2) \le 1, \quad \text{for } x \ne 0,$$

we have

$$-x^2 \leq x^2\cos(1/x^2) \leq x^2, \quad \text{for } x \neq 0.$$

Hence

$$-x^2 \leq f(x) \leq x^2 \quad \text{for } x \in \mathbb{R}.$$

Thus if we take $g(x) = -x^2$ and $h(x) = x^2$ all three conditions of the Squeeze Rule hold, with $a = 0$ and $I = \mathbb{R}$.

Hence f is continuous at 0, by the Squeeze Rule.

Solution for Exercise 3.2.5

Let $f(x) = x^2$ and $g(x) = x^2 + 1$. Then $\lim_{x\to\infty} \frac{e^{x^2}}{e^{x^2+1}} = \lim_{x\to\infty} \frac{e^{x^2}}{ee^{x^2}} = \frac{1}{e} \neq 1.$

Algebra

If you have problems attempting the questions in this section then you should consider studying M208 and/or M303

Solution for Exercise 3.3.1

(a) There are usually several ways of answering questions on group theory. Here is one way of approaching this one. We can show that a subgroup, H, is normal in a group G by working from the definition and showing that $g^{-1}Hg = H$ for all g in G. In other words we can show that $g^{-1}hg$ is in H for all g in G and h in H. We will use the notation $[h, k]$ for the *commutator* $hkh^{-1}k^{-1}$ of h and k. Taking a typical element of H, say $hkh^{-1}k^{-1}$ and conjugating by g in G gives:

$$\begin{aligned} g^{-1}[h,k]g &= g^{-1}hkh^{-1}k^{-1}g, \\ &= g^{-1}hgg^{-1}kgg^{-1}h^{-1}gg^{-1}k^{-1}g, \\ &= (g^{-1}hg)(g^{-1}kg)(g^{-1}h^{-1}g)(g^{-1}k^{-1}g), \\ &= [g^{-1}hg, g^{-1}kg], \end{aligned}$$

as this is clearly in H we are done. Note that part of the trick with this question is in getting the notation correct.

(b) There are, of course, many answers to this one. We need to think about what it means for a group to have a trivial commutator subgroup.
The commutator subgroup contains all elements of the form $hkh^{-1}k^{-1}$ where h and k are in M. If this is to be trivial, then we must have $hk = kh$ for all h and k in M so we must have M being Abelian. Therefore any non-trivial Abelian group will give a solution.

(c) The above argument works both ways and so any non Abelian group will have a non trivial commutator subgroup.

(d) We need to think about what this quotient will 'look like'. As with all quotient groups the elements will be cosets gH, with the subgroup H acting as an identity. Now from (b) we know that an Abelian group has a trivial commutator subgroup and so if G is Abelian we know that $\frac{G}{H} \cong G$ is also Abelian. Therefore it makes sense to think about whether the quotient group is always Abelian. You should be able to see that it will be. For a proof we have: For any h and k in G we have:

$$\begin{aligned} hHkHh^{-1}Hk^{-1}H &= hkh^{-1}k^{-1}H, \\ &= H. \end{aligned}$$

Therefore $hHkH = kHhH$.

Solution for Exercise 3.3.2

We need to check that the group axioms hold. These are:

(G1) Our set is closed.
(G2) We have an identity.
(G3) Every element has an inverse.
(G4) The operation is associative.

There are various way of showing this as there are various ways of writing the roots of unity. We will use the exponential form as this is the easiest to work with here.
The n^{th} roots of unity are $\{e^{\frac{2\pi ik}{n}} : n = 0 \ldots n-1\}$. Writing ω for $e^{\frac{2\pi i}{n}}$ we can now write the roots as $\{1, \omega^k : k = 1, \ldots n-1\}$
For (G1) we note that $\omega^j \times \omega^k = \omega^m$, where $m = (j+k) \bmod n$.
For (G2) we note that 1 is clearly an identity.
For (G3) we note that $(\omega^j)^{-1} = \omega^{n-j}$
and (G4) follows from the associativity of multiplication in $\mathbb{C}$

Working with Definitions

If you have problems attempting the questions in this section then you should consider studying M208 and/or M303.

Solution for Exercise 3.4.1

(a) As we are starting with an arbitrary set, X, we know that we are looking for a very general solution. The solution will be a collection of subsets. From (T1) we know that this collection must include X and $\emptyset$. Therefore we will consider $T = \{X, \emptyset\}$ to see whether this gives a topology on X.
Noting that $X \cap \emptyset = \emptyset \in T$ and $X \cup \emptyset = X \in T$ gives the result.
(Note that we do need to say $X \cup \emptyset = X \in T$ here and not $X \cup \emptyset = X \subseteq X$.)

(b) (T3′) only deals with the intersection of 2 sets. We can extend this to three or more sets as if A, B and C are in T then so is $A \cup B$ and therefore $(A \cup B) \cup C$. However this will only give us finite unions whereas (T3) also covers infinite unions.

Solution for Exercise 3.4.2

(a) We have $T = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and we need to check that (T1), (T2) and (T3) are satisfied. Clearly (T1) is. The easiest way to check (T2) and (T3) is to draw up a quick table of the unions and intersections and simply look to check that they are all in T.

Checking (T2)

$\cap$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a, b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$

clearly all of these are in T and so (T2) is satisfied.

Checking (T3)

$\cup$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

clearly all of these are in T and so (T3) is satisfied.

Note that we can just check that $A \cup B$ is in T for all A and B in T here because we only have finitely many sets.

(b) Here we simply note that $\{a, c\} \cap \{a, b\}$ is not in T' and so (T2) fails.

Solution for Exercise 3.4.3

(a) $C[0, 1]$ is the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ and so it isn't possible to compose them. The function $f : x \to 2x$ provides a counterexample to demonstrate this.

(b) As we now have the set of functions from $\mathbb{R}$ to $\mathbb{R}$ we can compose them. The composition of two continuous functions is continuous (note that this is a result that we are using here) so (M1) is satisfied. Composition of functions is clearly associative so we have (M2) and the identity function $e : x \to x$, (which is clearly continuous) acts as an identity.

Solution for Exercise 3.4.4
With this question you simply need to play around with the grid until you get something which works (a little like playing Sudoku). One solution is below:

1	1	1	1
1	-1	1	-1
1	i	-1	$-i$
1	$-i$	-1	i

(This question is related to the methods used to find some Dirichlet Characters, which are covered in M823.)