Science: A Level 3 Course
SMT359 Electromagnetism
SMT359 AYRF

The Open University

Are you ready for SMT359?

Contents

If, after working through these notes, you are still unsure about whether SMT359 is the right course for you, we advise you to seek further help and advice either from a Regional Advisor or from a Science Staff Tutor at your Regional Centre.

1 Introduction

If you are intending to study SMT359, you should make sure that you have the necessary background knowledge and skills to be able to enjoy the course fully and to give yourself the best possible chance of completing it successfully.

Read through these notes carefully and work through the questions. This is a useful exercise for all prospective students of the course, even for those who have already studied other Open University science and mathematics courses and who have completed the recommended prior study courses for SMT359 (see Section 2 below). Working through these diagnostic notes will serve as a reminder of some of the knowledge and skills which you will be assumed to have, either from OU Level 2 science and mathematics courses or from other prior study.

If you find that you can answer nearly all the questions in these notes, within about three hours and with only occasional reference to prerequisite material, it is likely that you are well prepared to start SMT359. If you have difficulties with some questions, or take much longer than three hours, this will indicate that you have gaps in your knowledge or that you need to improve your mathematical fluency.

Section 6 gives advice on specific remedial actions you can take. If you have substantial difficulties with five or more questions, you probably have a considerable amount of catching up to do, and should seriously assess whether SMT359 is the right course to attempt *at this stage of your studies.* Our experience is that students with insufficient preparation find Level 3 physics courses difficult, and often drop out.

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2 Suggested prior study

SMT359 is a Level 3 course which makes intellectual demands appropriate to the third year of a degree. The laws of electromagnetism, expressed in their most powerful and general form, involve the use of advanced mathematical techniques. Most of these techniques are introduced in the prerequisite mathematics course, MST209 *Mathematical methods and models*, (or its predecessor MST207 *Mathematical methods, models and modelling*). It is therefore *strongly* recommended that you have a good pass in MST209 (or MST207) and that you are familiar with the topics of complex numbers, vectors, vector calculus, partial differentiation, differential equations, multiple integrals and line integrals, all of which are taught or used in that course.

SMT359 also assumes that you have previously studied physics, especially mechanics and electromagnetism, at an introductory level. A good pass in S207 *The Physical World* (or its predecessor S271 *Discovering Physics*) is therefore recommended.

3 Mathematical skills

Mathematics is a vital tool in physics — it provides the language in which ideas are expressed and gives methods that allow exact conclusions to be reached. You will therefore need to be fluent with algebraic manipulation, vectors, differentiation and integration. In addition, electromagnetism makes use of some special mathematical techniques including vector calculus, multiple and line integrals and differential equations. All these topics are covered in the prerequisite mathematics course MST209 *Mathematical methods and models* (or its predecessor MST207) or are developed as part of SMT359.

The following questions test your understanding of some mathematical techniques that it is assumed you will be able to use at the outset of your studies of SMT359. The answers may be found in Section 8 of this booklet.

Vectors

Q1 (*geometric vectors; unit vectors; components and magnitude of a vector*) Given the vector $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are Cartesian unit vectors, determine

(a) the value of r_y, the y-component of $\mathbf{r}$,

(b) the magnitude $|\mathbf{r}|$ of vector $\mathbf{r}$,

(c) a unit vector $\widehat{\mathbf{r}}$ parallel to $\mathbf{r}$.

Q2 (*geometric vectors*) Evaluate the expression, $(\mathbf{r} - \mathbf{r}')f(\mathbf{r})/|\mathbf{r} - \mathbf{r}'|^3$, where $f(\mathbf{r}) = |\mathbf{r}|^2$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{r}' = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

Q3 (*scaling and adding vectors; dot product; cross product*) Given the vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$, determine

(a) $-\mathbf{a}$,

(b) $\mathbf{a} - 2\mathbf{b}$,

(c) $\mathbf{a} \cdot \mathbf{b}$,

(d) $\mathbf{a} \times \mathbf{b}$,

(e) $\mathbf{b} \times \mathbf{a}$.

Algebra, complex numbers and coordinates

Q4 (*trigonometry*)

Given that $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin^2 x + \cos^2 x = 1$, show that $(\sin 2\theta)^2 + 4 \sin^4 \theta = 2(1 - \cos 2\theta)$.

Q5 (*complex numbers, modulus and polar form*) Given $z = 3 - 2\mathrm{i}$ (where $\mathrm{i} = \sqrt{-1}$) write down:

(a) the imaginary part of z,

(b) the modulus $|z|$ of z,

(c) z^2.

(d) Express z in polar form.

Q6 (*complex numbers, Euler's formula*) Given $y = \mathrm{e}^{\mathrm{i}kx}$, where k and x are real, determine

(a) the real part of y,

(b) $|y|^2$.

Q7 (*cylindrical coordinates*) Give the cylindrical coordinates (r, ϕ, z) of the following points specified by Cartesian coordinates (x, y, z):

(a) $(0, 1, 0)$

(b) $(1, 1, 1)$.

Q8 (*spherical coordinates*) A potential energy function is given by

$$V(x, y, z) = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2).$$

Express this function in spherical coordinates.
For what values of B and C does the force depend only on the distance from the origin?

Q9 (*three-by-three determinants*) Evaluate the determinant,

$$\begin{vmatrix} a & b & c \\ x & 0 & -x \\ y & z & y \end{vmatrix}$$

Q10 (*summation symbol and factorial notation*) Evaluate

(a) $\displaystyle\sum_{j=1}^{3} j$

(b) $\displaystyle\sum_{n=1}^{4} (n^2+1)$

(c) $5!$

Calculus

Q11 (*differentiation of functions of a single variable: products, quotients and composite functions*) Differentiate the following

(a) $y = x \sin x$,
(b) $f(x) = (\tan x)/x$,

(c) $u = \mathrm{e}^{\mathrm{i}\omega t}$, where $\mathrm{i} = \sqrt{-1}$ and ω is constant,

(d) $y = \log_\mathrm{e}\left(1 + x^2\right)$.

Q12 (*properties of linear differential equations*)

(a) Find the values of k for which $\cos kx$ and $\sin kx$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 0.$$

(b) Write down an arbitrary linear combination of the solutions found in part (a). What property of the differential equation guarantees that your linear combination is a solution?

Q13 (*partial differentiation*) If $f(x,y) = cx/(x^2 + y^2)$, where c is a constant, determine $\partial f/\partial x$, $\partial f/\partial y$ and $\partial^2 f/\partial x \partial y$.

Q14 (*integration*) Evaluate $\displaystyle\int_a^b \frac{C}{r}\,\mathrm{d}r$, where C is a constant.

Q15 (*integration by substitution*) Given that

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \text{ determine } \int_0^2 \frac{4\,\mathrm{d}y}{1+9y^2}.$$

Q16 (*grad, div and curl*)

(a) If $f = x^2 + 2y + 3yz$, evaluate $\mathrm{grad}\, f$.

(b) If $\mathbf{A} = x^2\mathbf{i} + (2+y)\mathbf{j} + xy\mathbf{k}$, evaluate $\mathrm{div}\,\mathbf{A}$ and $\mathrm{curl}\,\mathbf{A}$.

Q17 (*multiple integrals*) Evaluate
$$\int_{y=0}^{y=2}\int_{x=0}^{x=1} x^2 y\,\mathrm{d}x\,\mathrm{d}y.$$

Q18 (*volume integrals*) Evaluate the volume integral $I = \displaystyle\int_B U_0 r^2\,\mathrm{d}V$ using spherical coordinates, where U_0 is a constant and B is a sphere of radius R centred on the origin.

Q19 (*vectors and line integrals*) Evaluate the line integral $\displaystyle\int_C \mathbf{E}\cdot\mathrm{d}\mathbf{l}$, where $\mathbf{E} = E(r)\mathbf{e}_\theta$, and $\mathbf{e}_\theta$ is a unit vector in plane polar coordinates in the direction of increasing θ, and C is a circular contour of radius R centred at the origin and which is traversed in the direction of increasing θ.

4 Key concepts in physics

SMT359 presents electromagnetism as a coherent body of knowledge, developed from basic principles explained fully within the course. Nevertheless, previous study of electromagnetism at the level of S207 is extremely helpful. In particular, a basic understanding of electrostatics, electrostatic potentials, electric and magnetic fields, field lines and equipotentials, the Lorentz force law, circuit analysis, electromagnetic induction and waves will provide a secure base for your studies.

The course also assumes a basic knowledge of mechanics, including Newton's laws, uniform acceleration and circular motion, work and energy, the force–potential energy relationship, the conservation laws of momentum and angular momentum, and properties of waves.

The following self-assessment questions test your understanding of some basic concepts in mechanics and electromagnetism. The answers may be found in Section 8 of this booklet.

Mechanics

Q20 (*Newton's second law; uniform acceleration*) A constant force of magnitude $2\times10^{-5}\,\mathrm{N}$ acts on a particle of mass $5\times10^{-6}\,\mathrm{kg}$.

(a) Determine the magnitude of the acceleration of the particle.

(b) If the particle is at rest at time $t = 0$, how far will it have moved by time $t = 1\,\mathrm{s}$?

Q21 (*momentum and kinetic energy*) Determine

(a) the magnitude of the momentum, and
(b) the kinetic energy

of a body of mass $3\,\mathrm{mg}$ moving at a speed of $5\,\mathrm{cm\,s^{-1}}$.

Q22 (*force and the gradient of potential energy*) A body moves along the x-axis in a region where its potential energy function is given by $V(x) = Cx^2$, where C is a constant. Determine the force acting on the body when it is at $x = 2\,\mathrm{m}$, given that $C = 5\,\mathrm{J\,m^{-2}}$.

Q23 (*centripetal acceleration, angular momentum, torque*) A body of mass 10^{-2} kg moves at a constant speed of $2\,\mathrm{m\,s^{-1}}$ along a circular path of radius 0.5 m.

(a) What are the magnitude and direction of the acceleration?

(b) Specify the force acting on the body.

(c) What is the torque of the force about the centre of the circular path?

(d) What is the magnitude of the angular momentum about the centre?

Electromagnetism and waves

$$\begin{aligned} \text{The elementary charge is } e &= 1.60 \times 10^{-19}\,\mathrm{C} \\ \text{The electron mass is } m &= 9.11 \times 10^{-31}\,\mathrm{kg} \\ (4\pi\epsilon_0)^{-1} &= 8.99 \times 10^{9}\,\mathrm{N\,m^2\,C^{-2}} \\ \text{The speed of light is } c &= 3.00 \times 10^{8}\,\mathrm{m\ s^{-1}} \\ \text{The electronvolt is } 1\,\mathrm{eV} &= 1.60 \times 10^{-19}\,\mathrm{J} \end{aligned}$$

Q24 (*electric current and charge*) What electric current is produced by a uniform beam of electrons moving from left to right if 10^{20} electrons pass a given point per second?

Q25 (*Lorentz force*) An electron moves with a speed of $1.0\,\mathrm{km\,s^{-1}}$ in a region where there is a uniform electric field of magnitude $3.0\,\mathrm{kV\,m^{-1}}$ and a uniform magnetic field of magnitude 2.0×10^{-3} T. Determine

(a) the magnitude of the electric force,

(b) the magnitude of the magnetic force, given that the electron is travelling perpendicular to the magnetic field direction.

Q26 (*the electronvolt*) Determine the kinetic energy in joules, and also in electronvolts, of

(a) an electron moving at a speed of $1000\,\mathrm{km\,s^{-1}}$,

(b) a proton that has been accelerated from rest through an electric potential difference of -5000 V.

Q27 (*Coulomb force law and electrostatic potential energy*) Determine (a) the magnitude of the electric force between two protons, and (b) their electrostatic potential energy, when they are separated by 1.0×10^{-11} m.

Q28 (*sinusoidal waves, wavelength, period, frequency*) Consider the wave described by $x = (0.2\,\mathrm{m})\cos(2.0z - 3.0t)$, where x and z are measured in metres and t in seconds. For this wave write down: (a) the amplitude, (b) the wavelength, (c) the angular frequency, (d) the period (e) the speed and direction of travel.

5 Other skills

You should have the following skills:

Study skills The ability to:

- organize time for study, and pace study appropriately;
- read effectively and identify relevant information from scientific texts and accounts;
- seek help when it is required.

Writing skills The ability to:

- write coherently;
- give succinct and complete definitions;
- write a scientific account with appropriate diagrams and equations.

Problem-solving and modelling skills The ability to:

- recognize the physical principles and equations that apply in described situations;
- translate a problem described in words into a form suitable for mathematical analysis;
- recognize information supplied implicitly or explicitly;
- draw appropriate diagrams;
- check answers and interpret a mathematical solution in physical terms.

ICT skills The ability to:

- use applications and simulation software;
- access information on the Web;
- communicate using email and conferencing software.

6 Suggested further reading and preparatory work

If the SAQ questions show that you need to learn more about certain topics or improve certain skills, the following sources are recommended.

Mathematical skills

The Open University Level 2 course MST209 is strongly recommended. The following Units are especially relevant for SMT359.

Units 2 and 3	on	differential equations
Unit 4	on	vectors
Unit 12	on	partial differentiation
Unit 24	on	vector calculus, line integrals
Unit 25	on	multiple integrals

If you need to refresh your knowledge of differentiation and integration, MST121 Block C and MS221 Block C are recommended. Complex numbers are covered in Unit D1 of MS221. Two extensive mathematical texts at an appropriate level to prepare for SMT359 are *Basic Mathematics for the Physical Sciences* and *Further Mathematics for the Physical Sciences*, both by R. Lambourne and M. Tinker.

Note that the final chapter of Book 1 of SMT359 is a *Mathematical Toolkit*, and if you think that the mathematics in the course might prove an obstacle to success, we recommend that you study this chapter before the scheduled start date for the course.

Key concepts in physics

The Open University Level 2 course S207 *The Physical World* is strongly recommended. Four of the books in this course are especially relevant for SMT359.

For introductory mechanics

Describing motion by R. Lambourne and A. Durrant

Predicting motion by R. Lambourne;

For introductory electromagnetism

Static fields and potentials by J. Manners

Dynamic fields and waves by A. Norton;

Many other physics textbooks describe the key physics concepts needed for SMT359. We recommend:

Fundamentals of physics by D. Halliday, R. Resnick and J. Walker

Physics by H. Ohanian

Physics for Scientists and Engineers by P. Tipler.

Other skills

Comprehensive guidance and advice on studying science courses may be found in *The Sciences Good Study Guide*. Basic skills are covered in Chapters 1, 2 and 6, and writing skills in Chapter 9.

Specific skills needed for SMT359 are also addressed in the Revision and Consolidation chapters of S207. In particular, Chapter 6 of *Predicting motion* and Chapter 5 of *Dynamic fields and waves* develop problem-solving skills. Chapter 5 of *Classical physics of matter* also develops the skills needed to give concise and full definitions.

7 References

Describing motion, by R. Lambourne and A. Durrant, Open University, 2007 (ISBN 978 0 7492 1913 0)

Predicting motion, by R. Lambourne, Open University, 2007 (ISBN 978 0 7492 1914 7)

Classical physics of matter, by J. Bolton, Open University, 2008 (ISBN 978 0 7492 1915 4)

Static fields and potentials. by J Manners, Open University, 2008 (ISBN 978 0 7492 1916 1)

Dynamic fields and waves, by A. Norton, Open University, 2008 (ISBN 978 0 7492 1917 8)

Fundamentals of physics, by D. Halliday, R. Resnick and J. Walker, John Wiley and Sons 2004 (ISBN 0 4714 65097)

Physics, by H. Ohanian, W. W. Norton & Company 2002 (ISBN 0 3939 4896 X)

Physics for Scientists and Engineers, by P. Tipler, Worth Publishers 1998 (ISBN 1 5725 9673 2)

Basic Mathematics for the Physical Sciences, by R Lambourne and M. Tinker, John Wiley and Sons 2000 (ISBN 0 4718 5207 4).

Further Mathematics for the Physical Sciences, by M. Tinker and R. Lambourne, John Wiley and Sons 2000 (ISBN 0 4718 6723 3).

The Sciences Good Study Guide, by A. Northedge. J. Thomas, A. Lane and A. Peasgood, Open University 1997 (ISBN 0 7492 3411 3)

8 Answers to self-assessment questions

Vectors

Q1(a) $r_y = -2$.

(b) $|\mathbf{r}|^2 = 3^2 + (-2)^2 + 1^2$. Therefore $|\mathbf{r}| = 14^{1/2}$.

(c) $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}| = 14^{-1/2}\mathbf{r}$.

Q2 $f(\mathbf{r}) = |\mathbf{r}|^2 = 1 + 2^2 = 5$.
$\mathbf{r} - \mathbf{r}' = 3\mathbf{j} - 4\mathbf{k}$. Therefore

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{3^2 + 4^2} = 5$$

and

$$\begin{aligned}\frac{(\mathbf{r} - \mathbf{r}')f(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3} &= \frac{(3\mathbf{j} - 4\mathbf{k})5}{125} \\ &= 0.12\mathbf{j} - 0.16\mathbf{k}.\end{aligned}$$

Q3(a) $-\mathbf{a} = -3\mathbf{i} + \mathbf{k}$.

(b) $\mathbf{a} - 2\mathbf{b} = (3\mathbf{i} - \mathbf{k}) - 2(\mathbf{i} - \mathbf{j}) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(c) $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$
$= 3 \times 1 + 0 \times (-1) + (-1) \times 0 = 3$.

(d) $\mathbf{a} \times \mathbf{b} =$

$$\begin{aligned}&(a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k} \\ &= (0 \times 0 - (-1) \times (-1))\mathbf{i} + ((-1) \times 1 - 3 \times 0)\mathbf{j} \\ &+ (3 \times (-1) - 0 \times 1)\mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} - 3\mathbf{k}.\end{aligned}$$

(e) $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Algebra, complex numbers and coordinates

Q4 The left-hand side of the required relationship can be written as follows:

$$\begin{aligned}(\sin 2\theta)^2 + 4\sin^4\theta &= (2\sin\theta\cos\theta)^2 + 4\sin^4\theta \\ &= 4\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right) \\ &= 4\sin^2\theta.\end{aligned}$$

Now $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, so

$$4\sin^2\theta = 4\left(\frac{1 - \cos 2\theta}{2}\right) = 2(1 - \cos 2\theta),$$

which is equal to the right-hand side of the required relationship.

Q5(a) $\operatorname{Im} z = -2$.

(b) $|z| = (z^* z)^{1/2} = ((3 + 2\mathrm{i})(3 - 2\mathrm{i}))^{1/2}$
$= (9 + 6\mathrm{i} - 6\mathrm{i} + 4)^{1/2} = \sqrt{13}$.

(c) $z^2 = (3 - 2\mathrm{i})^2 = 9 - 12\mathrm{i} - 4 = 5 - 12\mathrm{i}$.

(d) Using the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r = |z| = \sqrt{13}$, and $\theta = \tan^{-1}(\operatorname{Im} z / \operatorname{Re} z) = \tan^{-1}(-2/3) = -34°$.

Q6(a) $y = \cos kx + \mathrm{i}\sin kx$. Therefore $\operatorname{Re} y = \cos kx$.

(b) $|y|^2 = y^* y = \mathrm{e}^{-\mathrm{i}kx} \times \mathrm{e}^{\mathrm{i}kx} = 1$.

Q7 Use $r = \sqrt{x^2 + y^2}$, $\phi = \cos^{-1}(x/r)$ for $y \geq 0$, and $z' = z$.

(a) $(0, 1, 0) \to (1, \pi/2, 0)$.

(b) $(1, 1, 1) \to (\sqrt{2}, \pi/4, 1)$.

Q8 Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, then
$V(r,\theta,\phi)$
$= \frac{1}{2}(Ar^2\sin^2\theta\cos^2\phi + Br^2\sin^2\theta\sin^2\phi + Cr^2\cos^2\theta)$
$= \frac{1}{2}r^2(A\sin^2\theta\cos^2\phi + B\sin^2\theta\sin^2\phi + C\cos^2\theta)$.
The force depends on r only when $C = B = A$ giving $V(r) = Ar^2/2$.
(Note: Strictly speaking $V(r,\theta,\phi)$ and $V(x,y,z)$ are different functions and so different symbols should be used. However, you will find this abuse of notation in many physics texts, including SMT359.)

Q9 The determinant is given by

$$\begin{aligned}\det &= a(0+xz) - b(xy+xy) + c(xz-0)\\ &= xza - 2xyb + xzc.\end{aligned}$$

Q10(a) $1+2+3=6$.

(b) $(1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)=34$.

(c) $5! = 5\times4\times3\times2\times1 = 120$.

Calculus

Q11(a) $\mathrm{d}y/\mathrm{d}x = 1\times\sin x + x\cos x$.

(b) $f'(x) = \left(x\sec^2 x - \tan x\right)/x^2$.

(c) $\mathrm{d}u/\mathrm{d}t = \mathrm{i}\omega e^{\mathrm{i}\omega t} = \mathrm{i}\omega u$.

(d) $\mathrm{d}y/\mathrm{d}x = 2x/\left(1+x^2\right)$.

Q12(a) For $y = \cos kx$, $\mathrm{d}y/\mathrm{d}x = (-k)\sin kx$, $\mathrm{d}^2y/\mathrm{d}x^2 = -k^2\cos kx$. Substituting into the equation gives $k^2 = 4$. Therefore $k = \pm2$. Similarly for $\sin kx$, we find $k = \pm 2$.

(b) The solutions in part (a) are $y = \cos 2x$ and $y = \pm\sin 2x$. An arbitrary linear combination of this is $y = \alpha\cos 2x + \beta\sin 2x$, where α and β are arbitrary constants. The fact that the equation is *linear* guarantees that this is a solution.

Q13 The required partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{c}{x^2+y^2} - \frac{2cx^2}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{-2cxy}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{-2cy}{(x^2+y^2)^2} + \frac{8cx^2y}{(x^2+y^2)^3}.$$

Q14 $\displaystyle\int_a^b C/r\,\mathrm{d}r = C\left[\log_e r\ \right]_a^b = C\log_e(b/a)$.

Q15 Make the substitutions $a = 1, x^2 = 9y^2$, and $\mathrm{d}x = 3\,\mathrm{d}y$. Then

$$\begin{aligned}\int_0^2 \frac{4\,\mathrm{d}y}{1+9y^2} &= \frac{4}{3}\times\int_0^6 \frac{\mathrm{d}x}{a^2+x^2}\\ &= \frac{4}{3}\times\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_{x=0}^{x=6} = \frac{4}{3}\tan^{-1}6.\end{aligned}$$

Q16 (a)

$$\begin{aligned}\operatorname{grad} f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\\ &= 2x\mathbf{i} + (2+3z)\mathbf{j} + 3y\mathbf{k}.\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\\ &= 2x + 1 + 0\\ &= 2x + 1.\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j}\\ &\quad + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}\\ &= (x-0)\mathbf{i} + (0-y)\mathbf{j} + (0-0)\mathbf{k}\\ &= x\mathbf{i} - y\mathbf{j}.\end{aligned}$$

Q17

$$\int_{y=0}^{y=2}\int_{x=0}^{x=1} x^2y\,\mathrm{d}x\,\mathrm{d}y = \int_{y=0}^{y=2} y\left[x^3/3\right]_0^1\mathrm{d}y$$
$$= \int_{y=0}^{y=2}(y/3)\,\mathrm{d}y = \left[y^2/6\right]_0^2 = 2/3.$$

Q18

$$\int_B U_0 r^2\,\mathrm{d}V =$$
$$U_0\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} r^2r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r$$
$$= U_0\left[\int_0^R r^4\,\mathrm{d}r\right]\left[\int_0^\pi \sin\theta\,\mathrm{d}\theta\right]\left[\int_0^{2\pi}\mathrm{d}\phi\right]$$
$$= U_0\left[\frac{r^5}{5}\right]_0^R\left[-\cos\theta\ \right]_0^\pi\left[\phi\ \right]_0^{2\pi} = U_0\times\frac{R^5}{5}\times2\times2\pi$$
$$= 4\pi U_0\frac{R^5}{5}.$$

Q19 For a circular contour, of radius R, $\mathrm{d}l = R\,\mathrm{d}\theta$, i.e. $\mathrm{d}\mathbf{l} = R\,\mathrm{d}\theta\mathbf{e}_\theta$, and θ is integrated from 0 to 2π. Then $\mathbf{E}\cdot\mathrm{d}\mathbf{l} = E(R)R\,\mathrm{d}\theta$ and $\displaystyle\int_0^{2\pi} E(R)R\,\mathrm{d}\theta = 2\pi E(R)R$.

Mechanics

Q20(a) Since $F = ma$, then

$$a = \left(2 \times 10^{-5}\,\text{N}\right) / \left(5 \times 10^{-6}\,\text{kg}\right) = 4\,\text{N}\,\text{kg}^{-1} = 4\,\text{m}\,\text{s}^{-2}.$$

(b) Use $s = ut + \frac{1}{2}at^2$ with $u = 0$. Then

$$s = \frac{1}{2} \times 4\ \text{m s}^{-2} \times (1\,\text{s})^2 = 2\,\text{m}.$$

Q21(a) Momentum $\mathbf{p} = m\mathbf{v}$, so

$$p = mv = 3 \times 10^{-6}\,\text{kg} \times 5 \times 10^{-2}\,\text{m}\,\text{s}^{-1} = 1.5 \times 10^{-7}\,\text{kg}\,\text{m}\,\text{s}^{-1}.$$

(b) Kinetic energy $= \frac{1}{2}mv^2$

$$= \frac{1}{2} \times 3 \times 10^{-6}\,\text{kg} \times \left(5 \times 10^{-2}\,\text{m}\,\text{s}^{-1}\right)^2 = 3.75 \times 10^{-9}\,\text{J} \approx 4 \times 10^{-9}\,\text{J}$$

to one significant figure.

Q22 Use $F_x = -\text{d}V/\text{d}x = -2Cx$. At $x = 2\,\text{m}$ and with $C = 5\,\text{J}\,\text{m}^{-2}$, this is $F_x = -2 \times 5\,\text{J}\,\text{m}^{-2} \times 2\,\text{m} = -20\,\text{N}$. Note: in three dimensions, the corresponding formula is

$$\mathbf{F} = -\text{grad}\,V = -\nabla V = -\mathbf{i}\,\partial V/\partial x - \mathbf{j}\,\partial V/\partial y - \mathbf{k}\,\partial V/\partial z.$$

Q23(a) The centripetal acceleration is of magnitude $v^2/r = \left(2\,\text{m}\,\text{s}^{-1}\right)^2/(0.5\,\text{m}) = 8\,\text{m}\,\text{s}^{-2}$. Its direction is towards the centre of the circular path.

(b) $\mathbf{F} = m\mathbf{a}$. Therefore $|\mathbf{F}| = 10^{-2}\,\text{kg} \times 8\,\text{m}\,\text{s}^{-2} = 8 \times 10^{-2}\,\text{N}$. Its direction is towards the centre.

(c) Torque is $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$, that is $|\boldsymbol{\Gamma}| = |\mathbf{r}||\mathbf{F}|\sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. But $\mathbf{r}$ and $\mathbf{F}$ are antiparallel (i.e. $\theta = 180°$). Therefore $\boldsymbol{\Gamma} = \mathbf{0}$.

(d) Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Therefore $|\mathbf{L}| = |\mathbf{r}||\mathbf{p}|\sin\theta$. For circular motion, $\mathbf{r}$ and $\mathbf{p}$ are perpendicular (i.e. $\theta = 90°$). Therefore

$$|\mathbf{L}| = |\mathbf{r}||\mathbf{p}| = mrv = 10^{-2}\,\text{kg} \times 0.5\,\text{m} \times 2\,\text{m}\,\text{s}^{-1} = 10^{-2}\,\text{kg}\,\text{m}^2\,\text{s}^{-1}.$$

Electromagnetism and waves

Q24 Magnitude of electric current = number of particles per unit time × magnitude of electric charge of particle

$$= 10^{20}\,\text{s}^{-1} \times 1.60 \times 10^{-19}\,\text{C} = 1.60 \times 10^{1}\,\text{C}\,\text{s}^{-1} = 16\,\text{A}.$$

Because the charge of an electron is $-e$, the current flows from right to left.

Q25 Lorentz force $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$.

(a) Magnitude of electric force is

$$qE = 1.60 \times 10^{-19}\,\text{C} \times 3.0 \times 10^{3}\,\text{V}\,\text{m}^{-1} = 4.8 \times 10^{-16}\,\text{N}.$$

(b) Magnitude of magnetic force is $q\mathbf{v} \times \mathbf{B}$

$$= qvB\sin\theta = qvB = 1.60 \times 10^{-19}\,\text{C} \times 1.0 \times 10^{3}\,\text{m}\,\text{s}^{-1} \times 2.0 \times 10^{-3}\,\text{T} = 3.2 \times 10^{-19}\,\text{N}.$$

Q26(a) Kinetic energy $= \frac{1}{2}mv^2$

$$= \frac{1}{2} \times 9.11 \times 10^{-31}\,\text{kg} \times \left(1.0 \times 10^{6}\,\text{m}\,\text{s}^{-1}\right)^2 = 4.56 \times 10^{-19}\,\text{J} = \frac{4.56 \times 10^{-19}\,\text{J}}{1.60 \times 10^{-19}\,\text{J}\,\text{eV}^{-1}} = 2.8\,\text{eV}.$$

(b) Kinetic energy acquired = potential energy lost

$$= -qV = -1.60 \times 10^{-19}\,\text{C} \times (-5000\,\text{V}) = 8.00 \times 10^{-16}\,\text{J} = \frac{8.00 \times 10^{-16}\,\text{J}}{1.60 \times 10^{-19}\text{J}\,\text{eV}^{-1}} = 5.0 \times 10^{3}\,\text{eV}.$$

Q27(a) Electrostatic or Coulomb force is

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} = 8.99 \times 10^{9}\,\text{N}\,\text{m}^2\,\text{C}^{-2} \times \frac{\left(1.60 \times 10^{-19}\,\text{C}\right)^2}{\left(1.0 \times 10^{-11}\,\text{m}\right)^2} = 2.3 \times 10^{-6}\,\text{N}.$$

(b) Potential energy is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} = 8.99 \times 10^{9}\,\text{N}\,\text{m}^2\,\text{C}^{-2} \times \frac{\left(1.60 \times 10^{-19}\,\text{C}\right)^2}{1.0 \times 10^{-11}\,\text{m}} = 2.3 \times 10^{-17}\,\text{J}.$$

Q28(a) Amplitude $= 0.2\,\text{m}$.

(b) Wavelength $= 2\pi/2.0\ \text{m} = 3.1\,\text{m}$.

(c) Angular frequency $= 3.0\,\text{rad}\,\text{s}^{-1}$.

(d) Period $= 2\pi/3.0\ \text{s} = 2.1\,\text{s}$.

(e) Wave moves in +ve z-direction with speed $3.0/2.0\,\text{m}\,\text{s}^{-1} = 1.5\,\text{m}\,\text{s}^{-1}$.