

2023 PhD Projects

Project title	Bouncing Queens and Morphic Games
Principal supervisor	Dan Rust
Second supervisors	Katherine Staden Gwyneth Stallard
Discipline	Pure mathematics
Research area/keywords	Combinatorics on Words, Combinatorial Game Theory, Aperiodic Order
Suitable for	Full time applicants, Part time applicants

Project background and description

In this project, we will investigate links between combinatorial game theory and the theory of morphisms on finite alphabets; how the generated sequences reveal winning strategies.

Alice and Bob want to play a game. Alice takes her trusty $n \times n$ chessboard out of her backpack and a single queen from a chess set. “*The rules are simple*” she tells Bob. “*You get to place the queen anywhere on the board and then we each take a turn moving the queen either down, left or diagonally down-and-left. I get to make the first move. Whoever gets the queen into the corner of the board is the winner.*” Bob quickly slams the queen down and, much to Alice’s annoyance, exclaims that he has already won. On which squares could Bob have placed the Queen?

It turns out that Bob could have placed the queen at any position (n, m) such that $n = \lfloor k\phi \rfloor$ and $m = \lfloor k\phi^2 \rfloor$ where ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$. These n and m also correspond to the positions of the k th A and B in the *Fibonacci word* $ABAABABAABAABAB \dots$, which is the fixed point of the morphism $A \mapsto AB, B \mapsto A$. This game is known as *Whythoff’s Nim*, the most famous example of a so-called *morphic game*. If the queen is allowed to bounce off of the boundary of the chess board, it turns out that this new game is also morphic, but now coded by the *Tribonacci morphism* $A \mapsto AB, B \mapsto AC, C \mapsto A$.

Does the Quadribonacci morphism $A \mapsto AB, B \mapsto AC, C \mapsto AD, D \mapsto A$ also code a morphic game? Which other queens on a chess board are morphic? Which morphisms code a game? These are all open problems and will be explored.

These questions are introduced in the recent article [1], written to be a light-hearted introduction to morphic queens. A good entry-point to Combinatorial Game Theory is Berlekamp, Conway and Guy’s famous textbook [2]. Duchêne and Rigo introduced the idea of a game being morphic in [3].

Background reading/references

- [1] R. Fokkink, G. F. Ortega, and D. Rust, *Corner the Empress*, 2021. preprint available at <https://arxiv.org/abs/2204.11805>.
- [2] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*. Vol. 1. Second edition. A K Peters, Ltd., Natick, MA, 2001.
- [3] E. Duchêne and M. Rigo, *A morphic approach to combinatorial games: the Tribonacci case*,

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